Tutorial:
General Introduction to Fault-tolerant Quantum Computing

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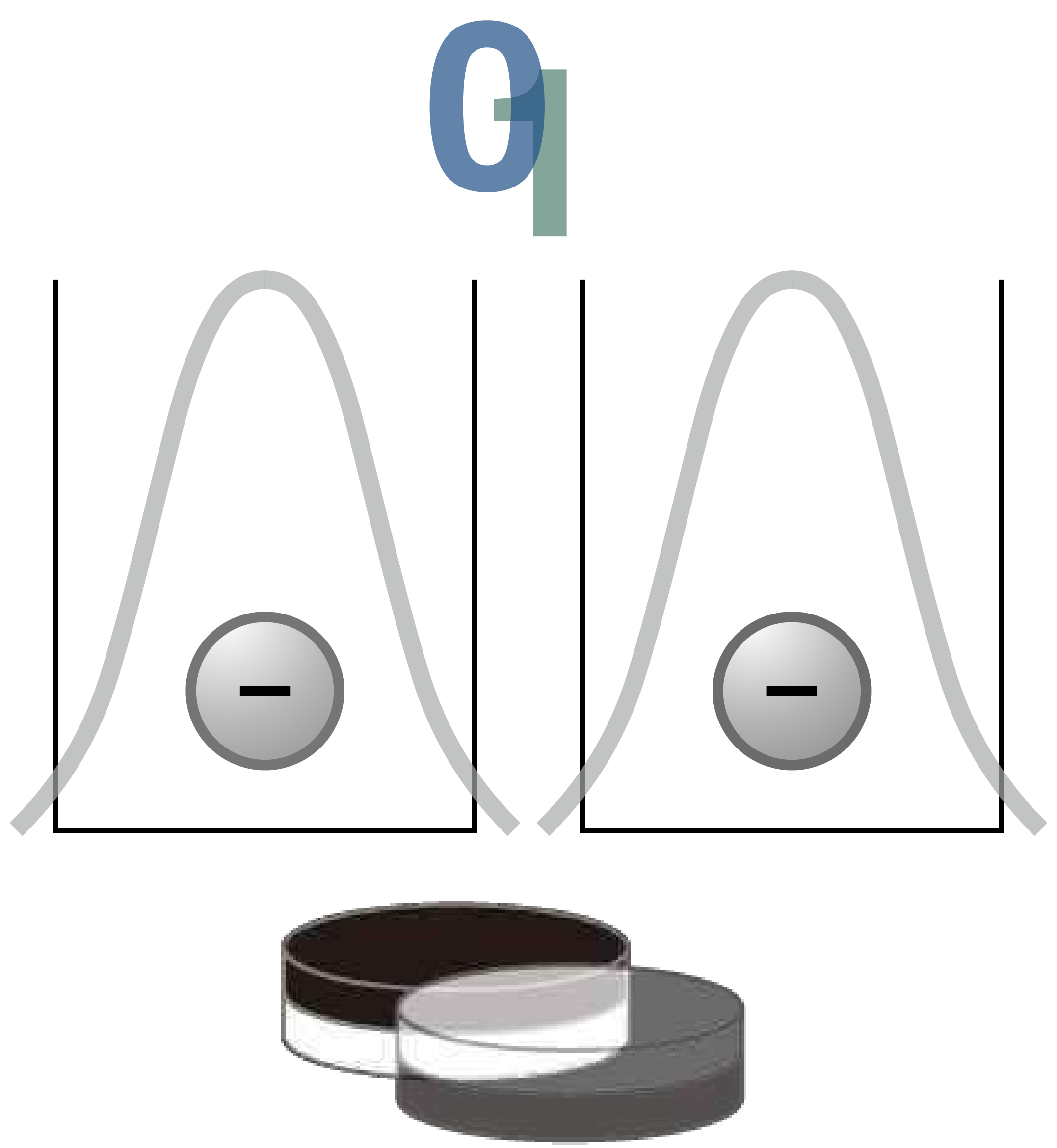
RIKEN Center for Quantum Computing

Why quantum error correction?

Quantum Fragility: Don’t watch it to see it.

Qubit

superposed state



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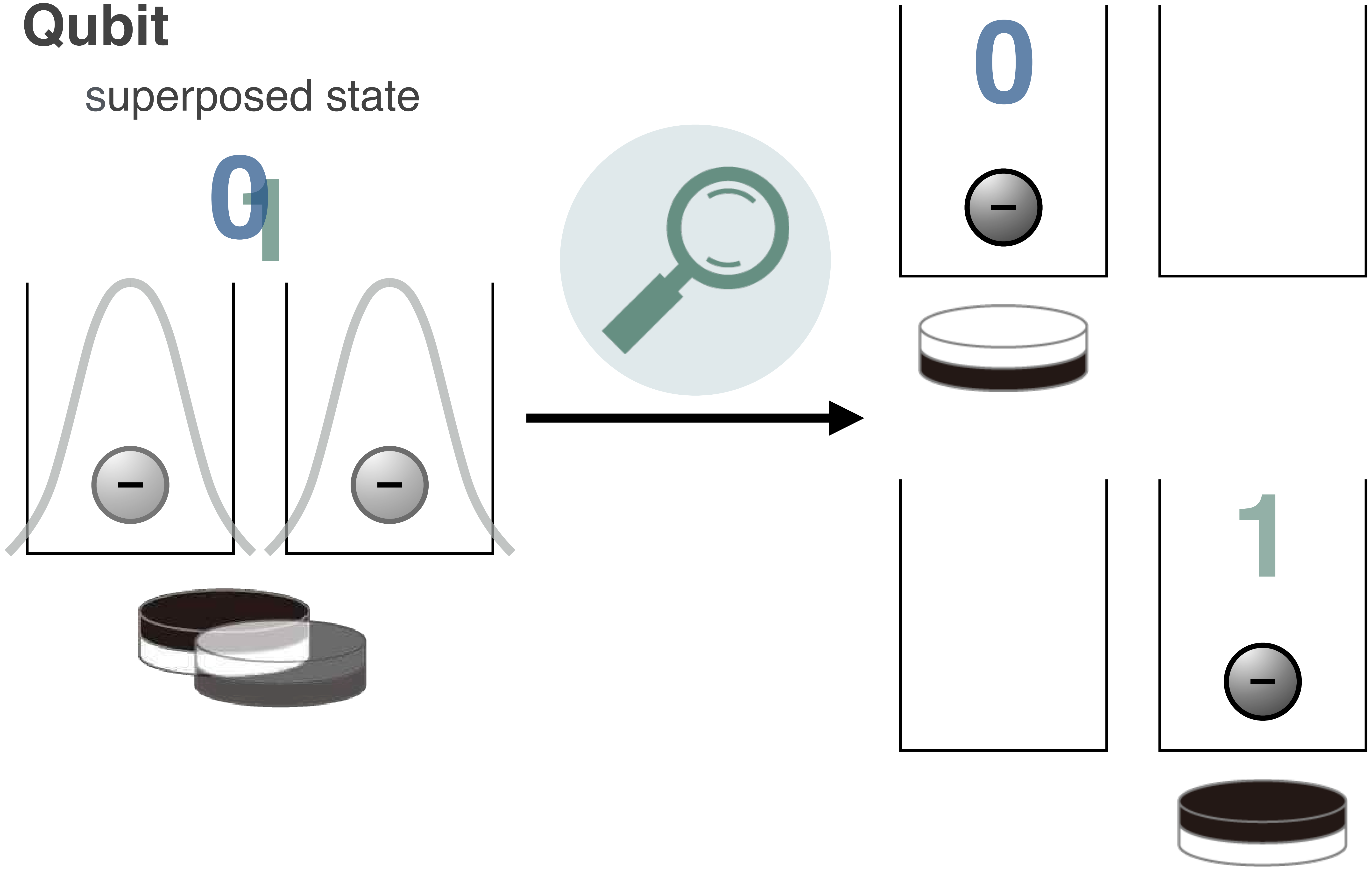

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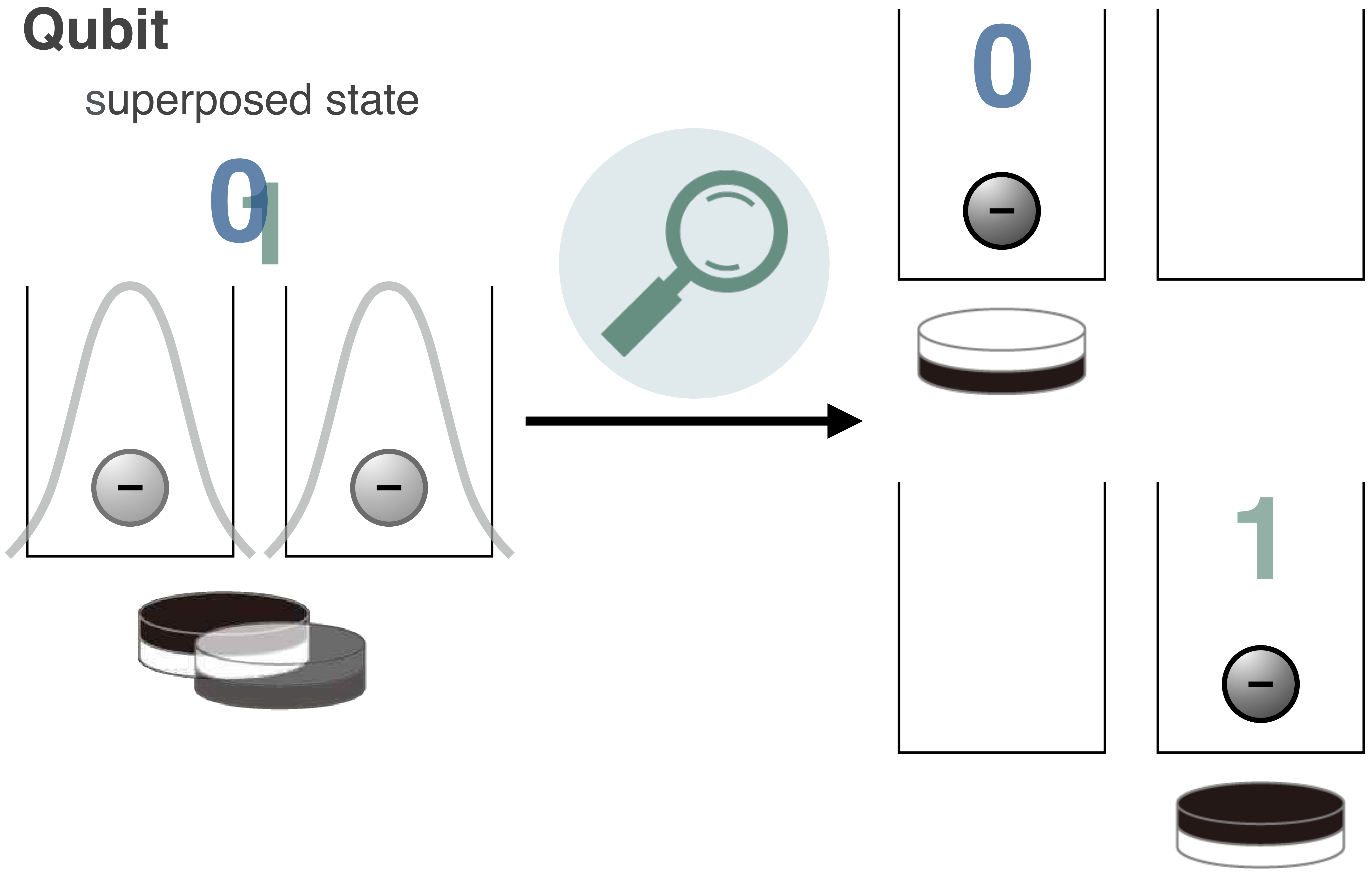
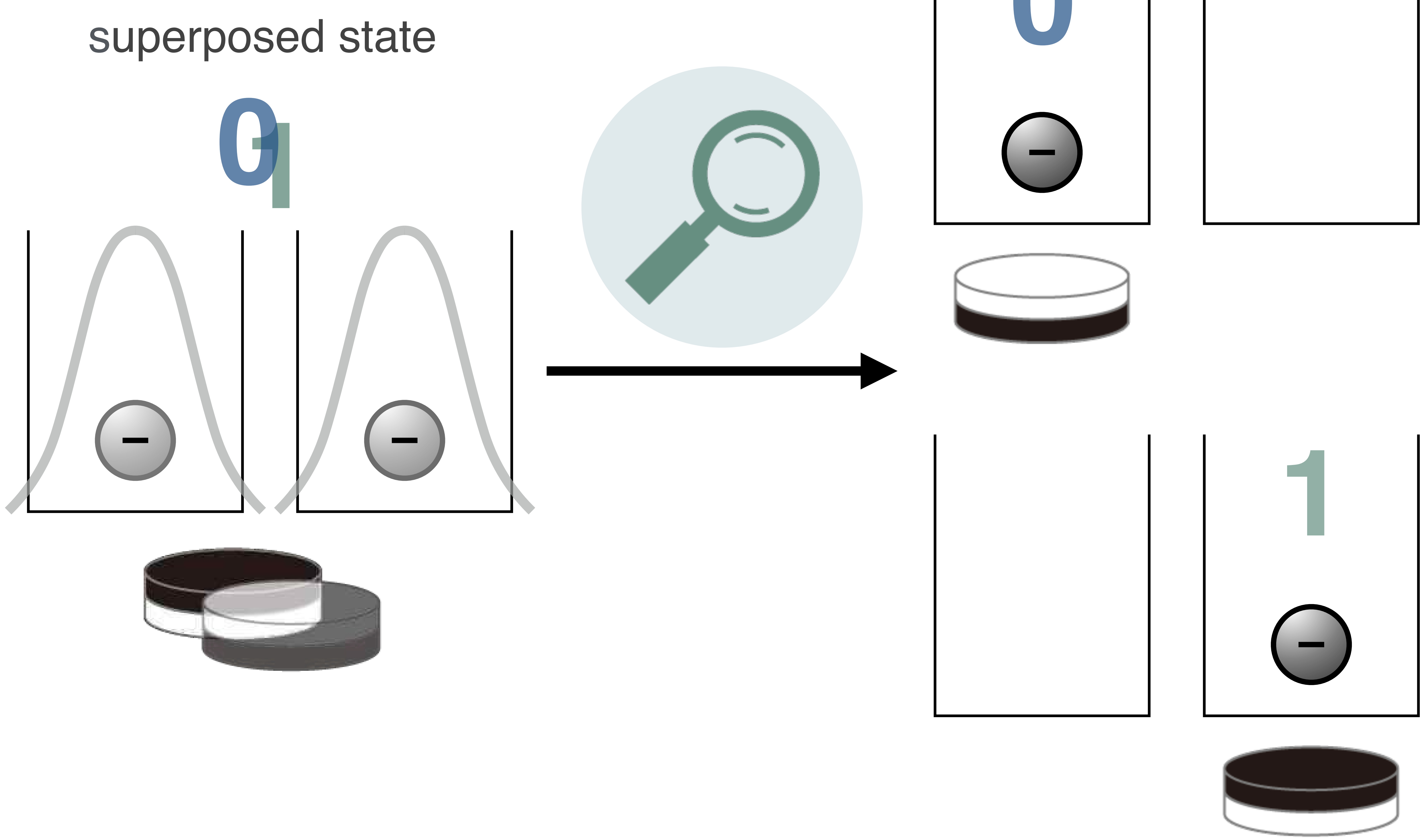


Undesirable interactions with environmental systems cause qubits to be observed.

→ **Decoherence**

Quantum Fragility: Don't watch it to see it.

Qubit



Undesirable interactions with environmental systems cause qubits to be observed.

→ **Decoherence**

Quantum information

→ Exponentially many complex amplitudes.

→ Suffering from analog noise.

No-cloning theorem

→ Quantum state cannot be copied.

[Wootters-Zurek82]

Quantum Fragility: Don't watch it to see it.

Qubit

Undesirable interactions with

All papers on quantum computing should carry a footnote: *"This proposal, like all proposals for quantum computation, relies on* ***speculative technology****, does not in its current form take into account all possible sources of noise, unreliability and manufacturing error, and* ***probably will not work****."*

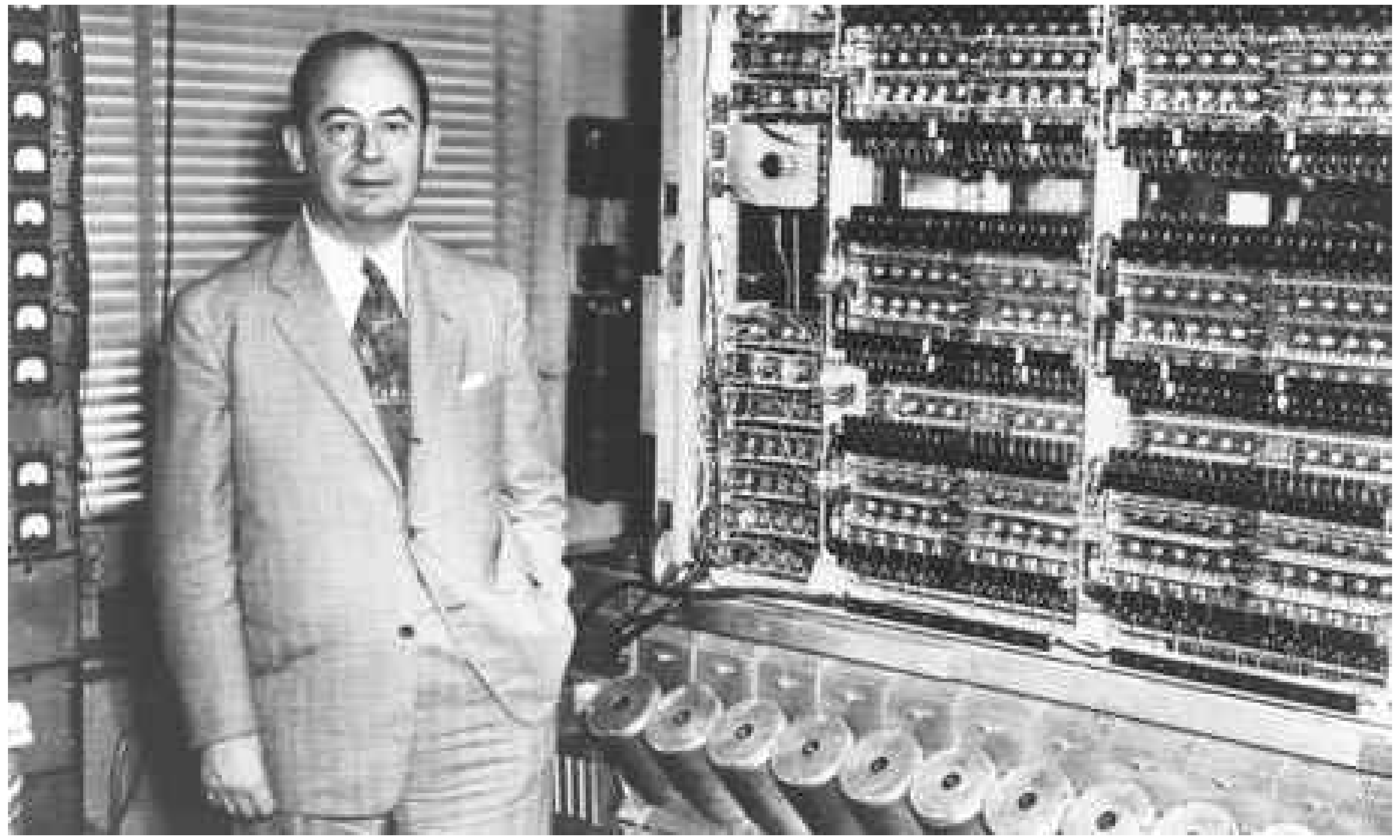
S. Lloyd Nature 400 720 (1999)
Rolf Landauer @IBM

→ Quantum state cannot be copied.
[Wootters-Zurek82]

History of Reliable Computing

History of Reliable Computing

EDVAC: Electric discrete variable automatic computer



John von Neumann with the stored-program computer at the Institute for Advanced Study, Princeton, New Jersey, in 1945. Photograph: Getty

History of Reliable Computing

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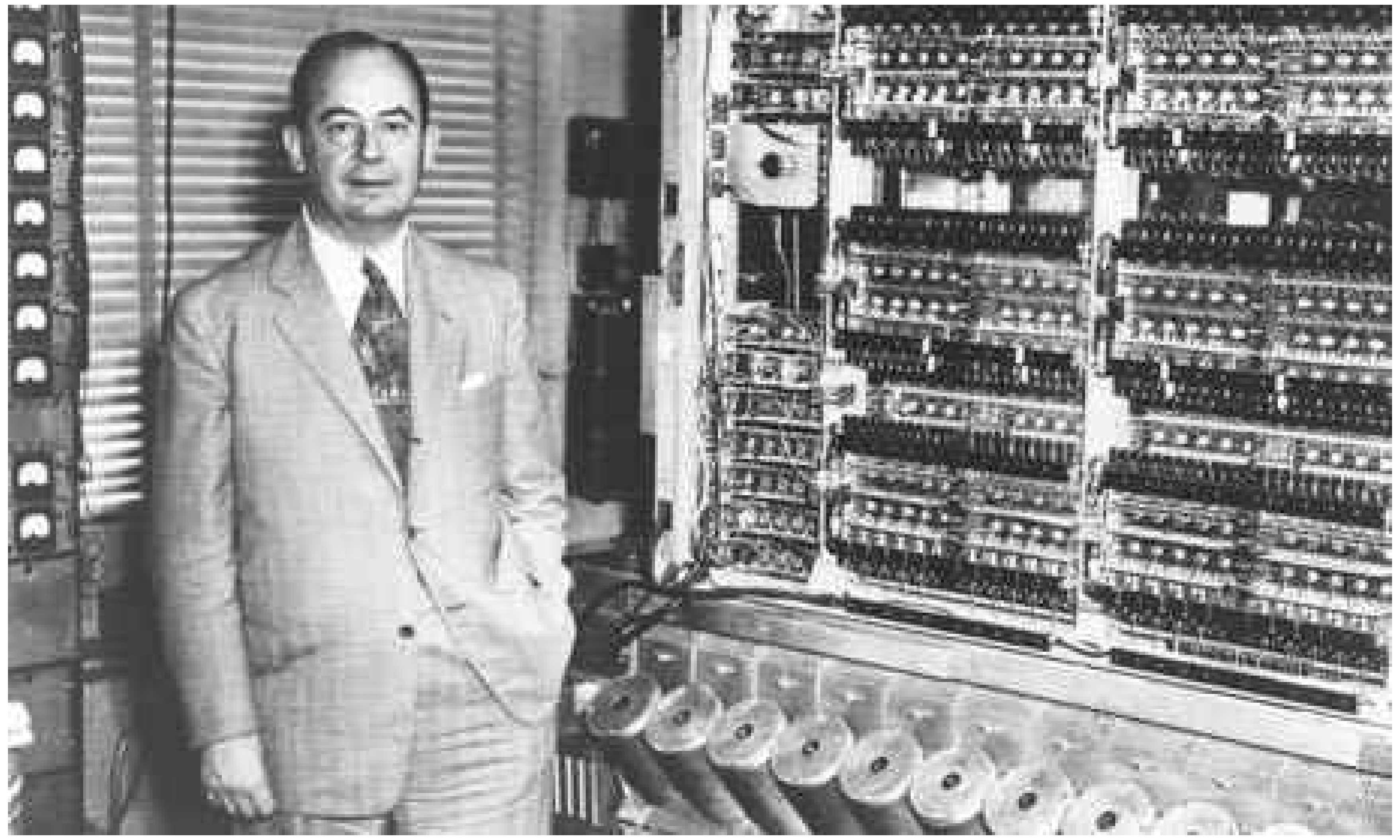

John von Neumann with the stored-program computer at the Institute for Advanced Study, Princeton, New Jersey, in 1945. Photograph: Getty

For the recognition and correction of such malfunctions ***intelligent human intervention will in general be necessary***.

by J. von Neumann (1945)

"First draft of a report on the EDVAC"

History of Reliable Computing

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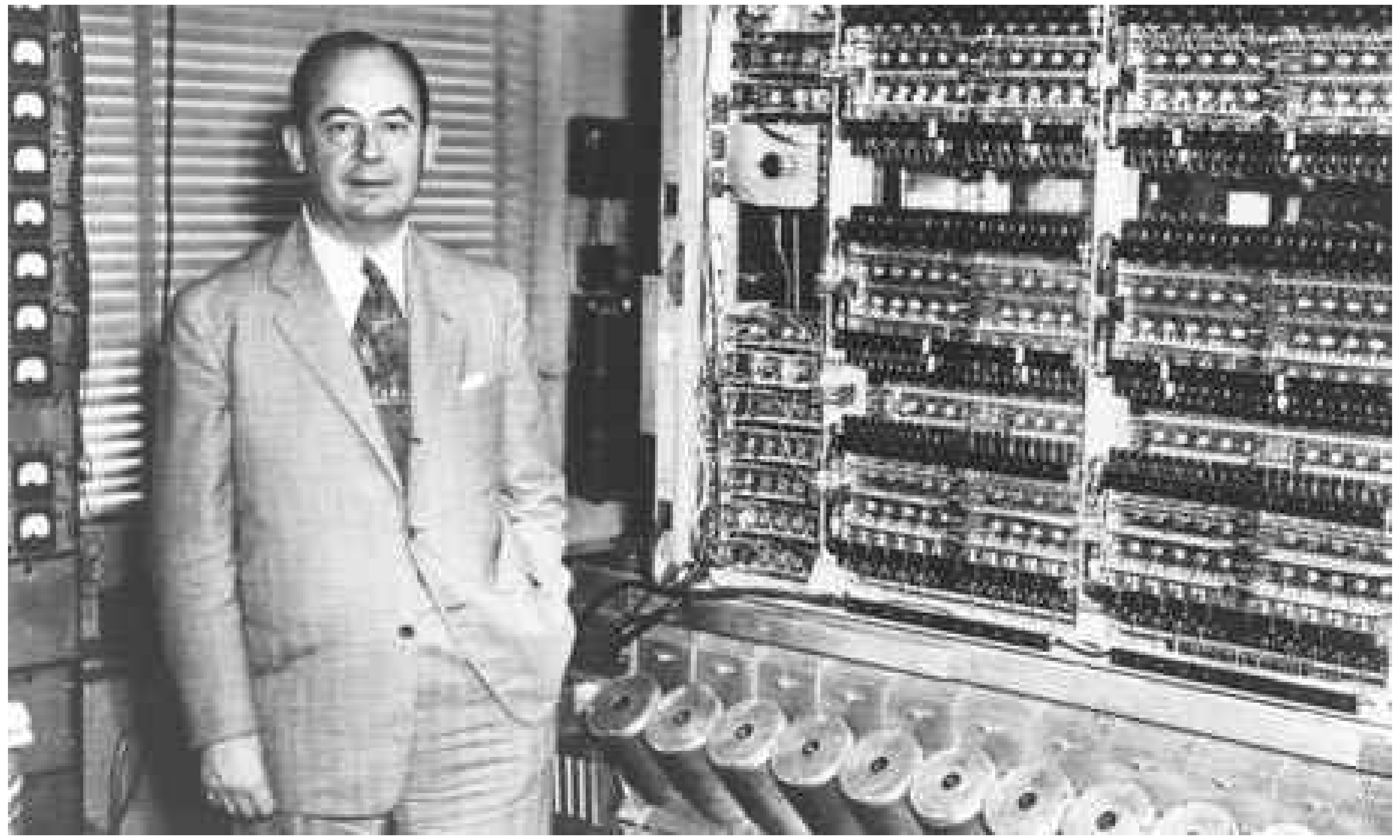

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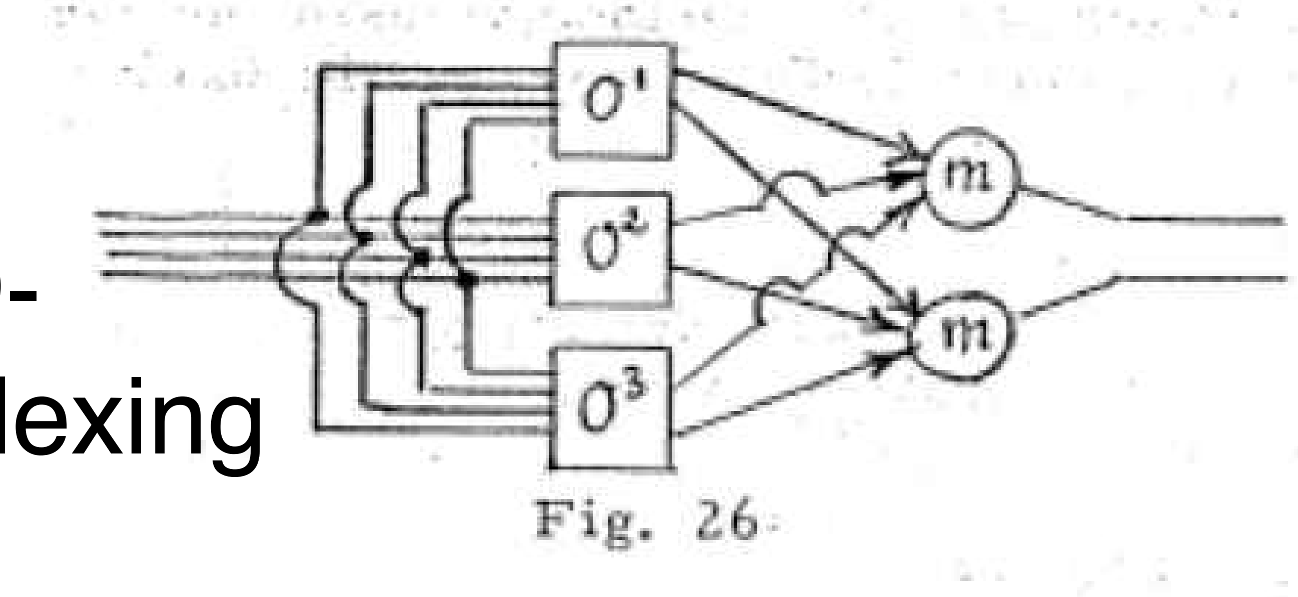
"First draft of a report on the EDVAC"

about ten years later

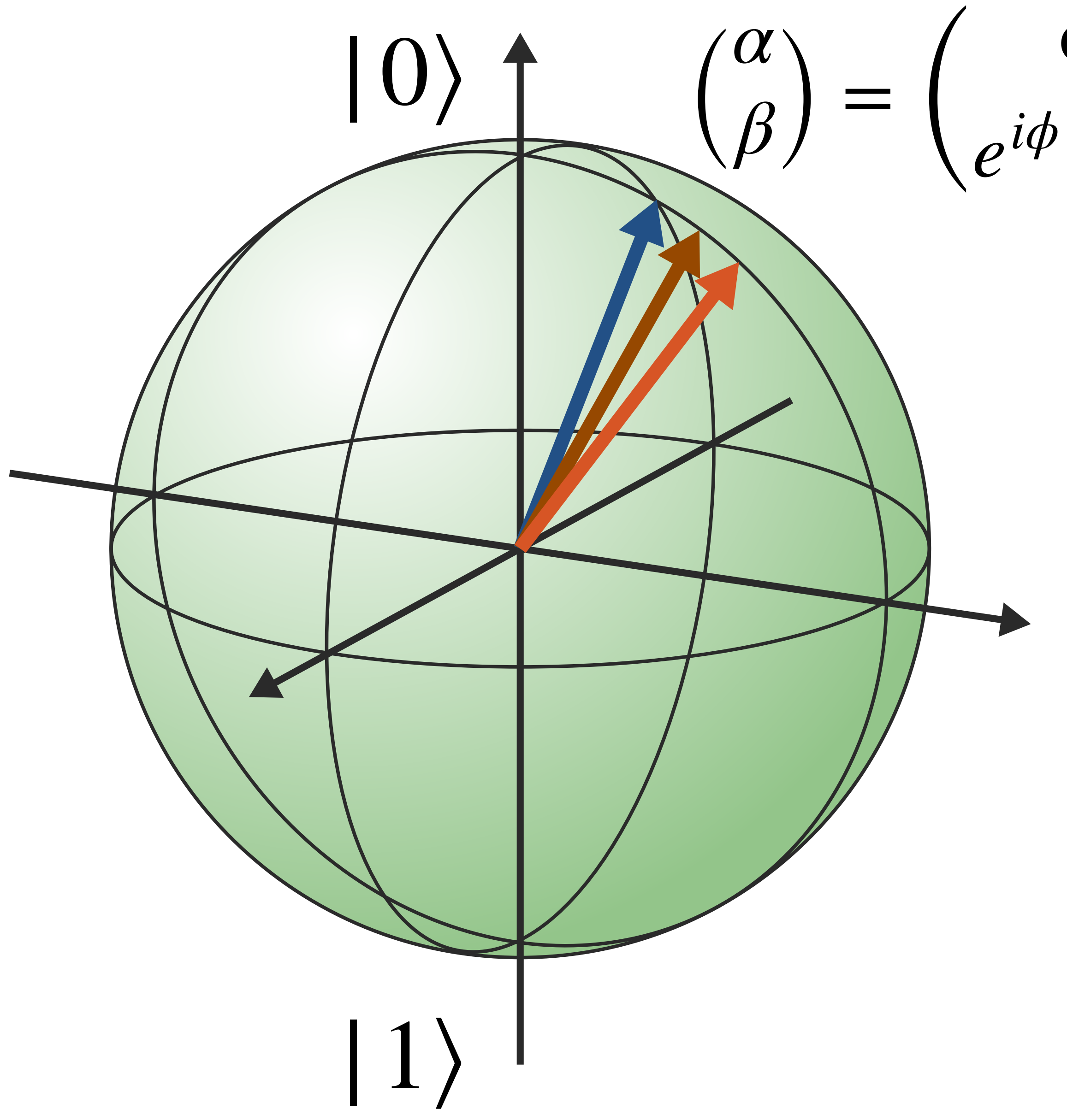
NAND-multiplexing



J. von Neumann (1956)

"Probabilistic logics and the synthesis of reliable organisms from unreliable components"

Quantum Error Correction

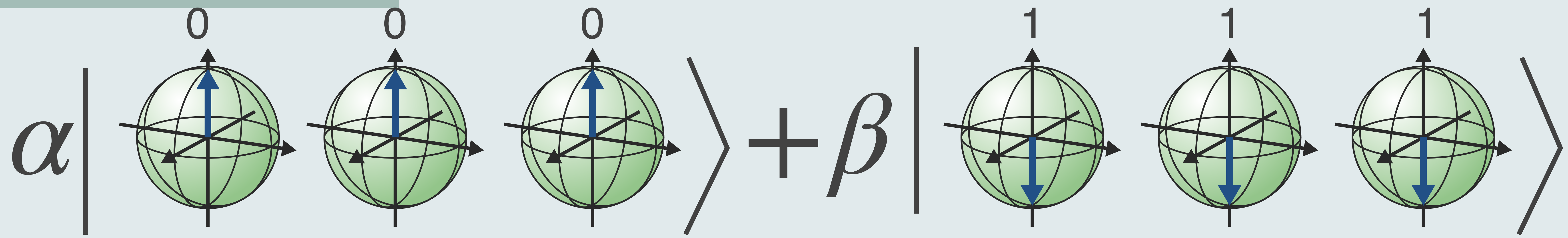


$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\cos(\theta/2) \\ e^{i\phi}\sin(\theta/2)\end{pmatrix}$$

The state of the qubit is unintentionally shifted because of entanglement with environment.

→fight entanglement with entanglement (by J. Preskill)

Quantum Error Correction



Shor, Peter W. "Scheme for reducing decoherence in quantum computer memory." Physical review A 52.4 (1995): R2493.

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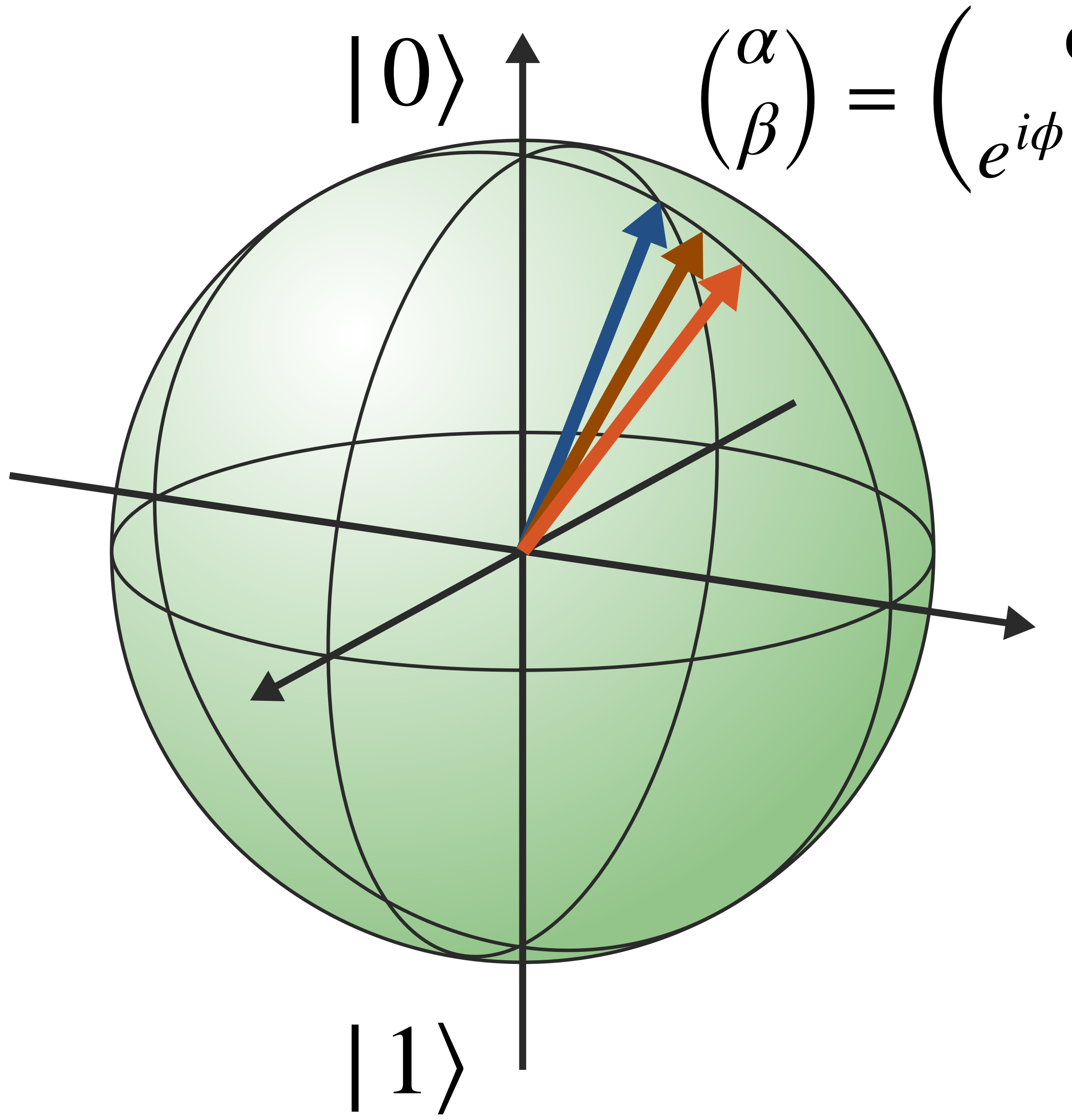

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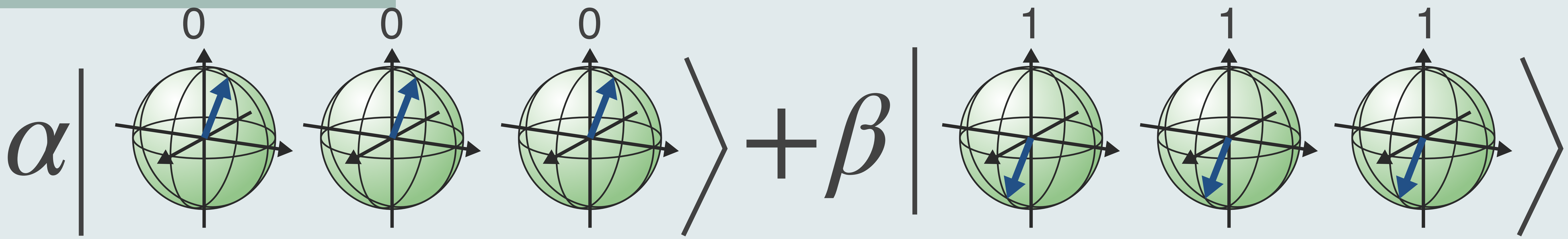

Individual qubits are affected by analog errors due to the noise.

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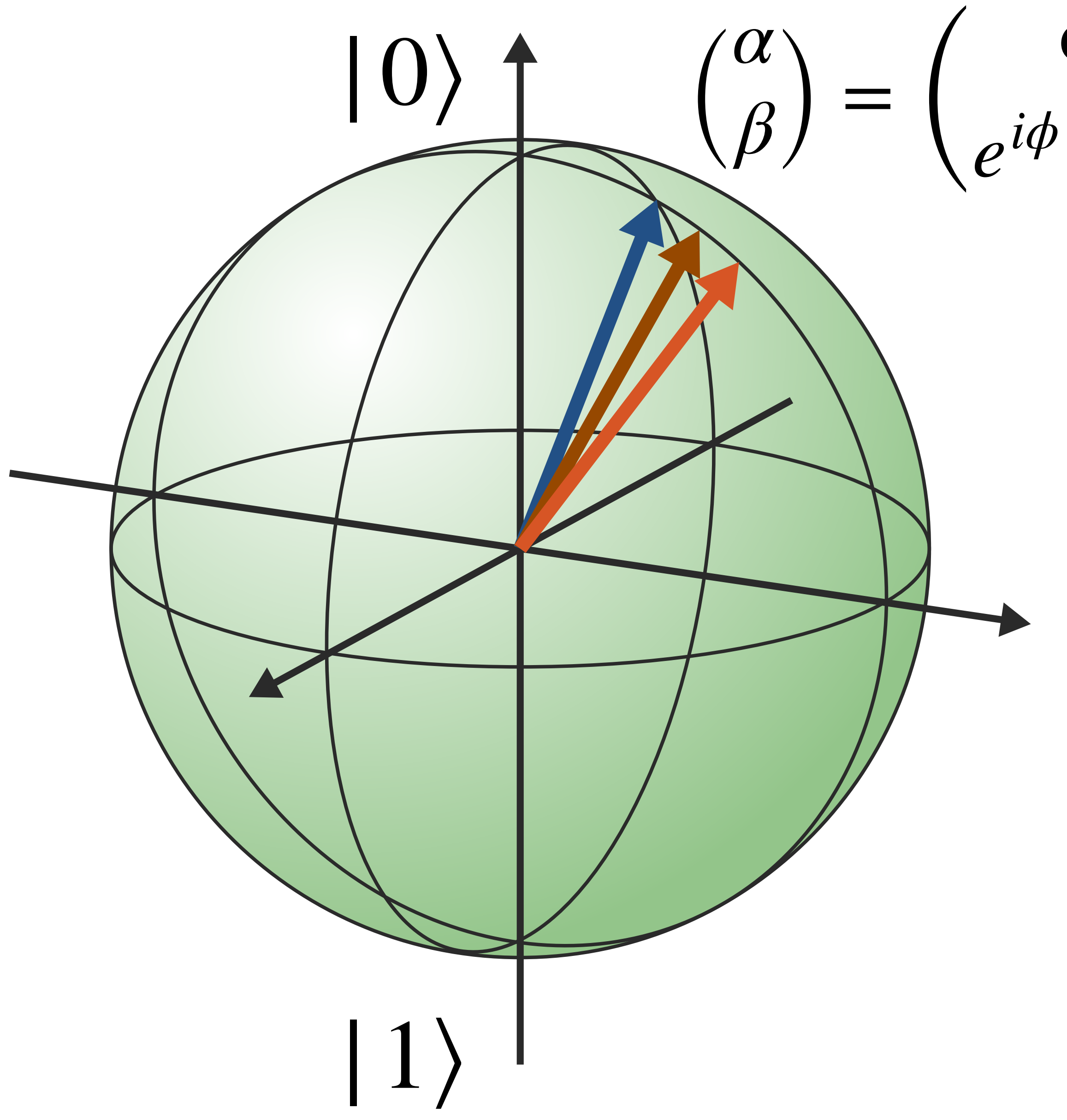

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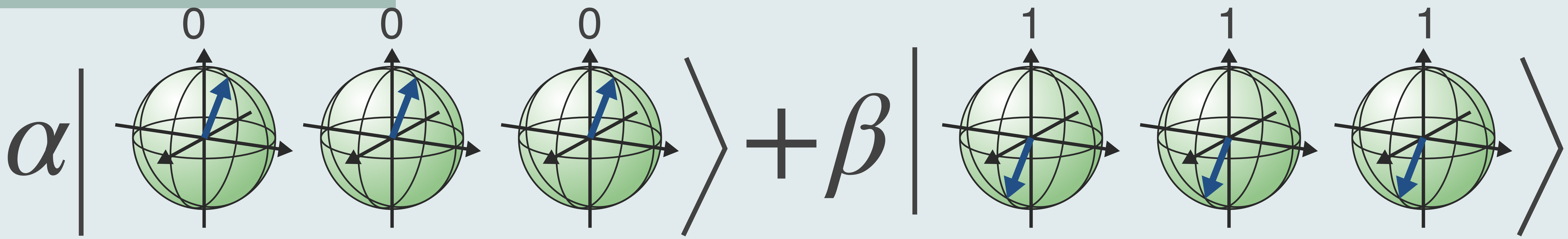
Quantum Error Correction

Do measurement what "digital" error occurs.



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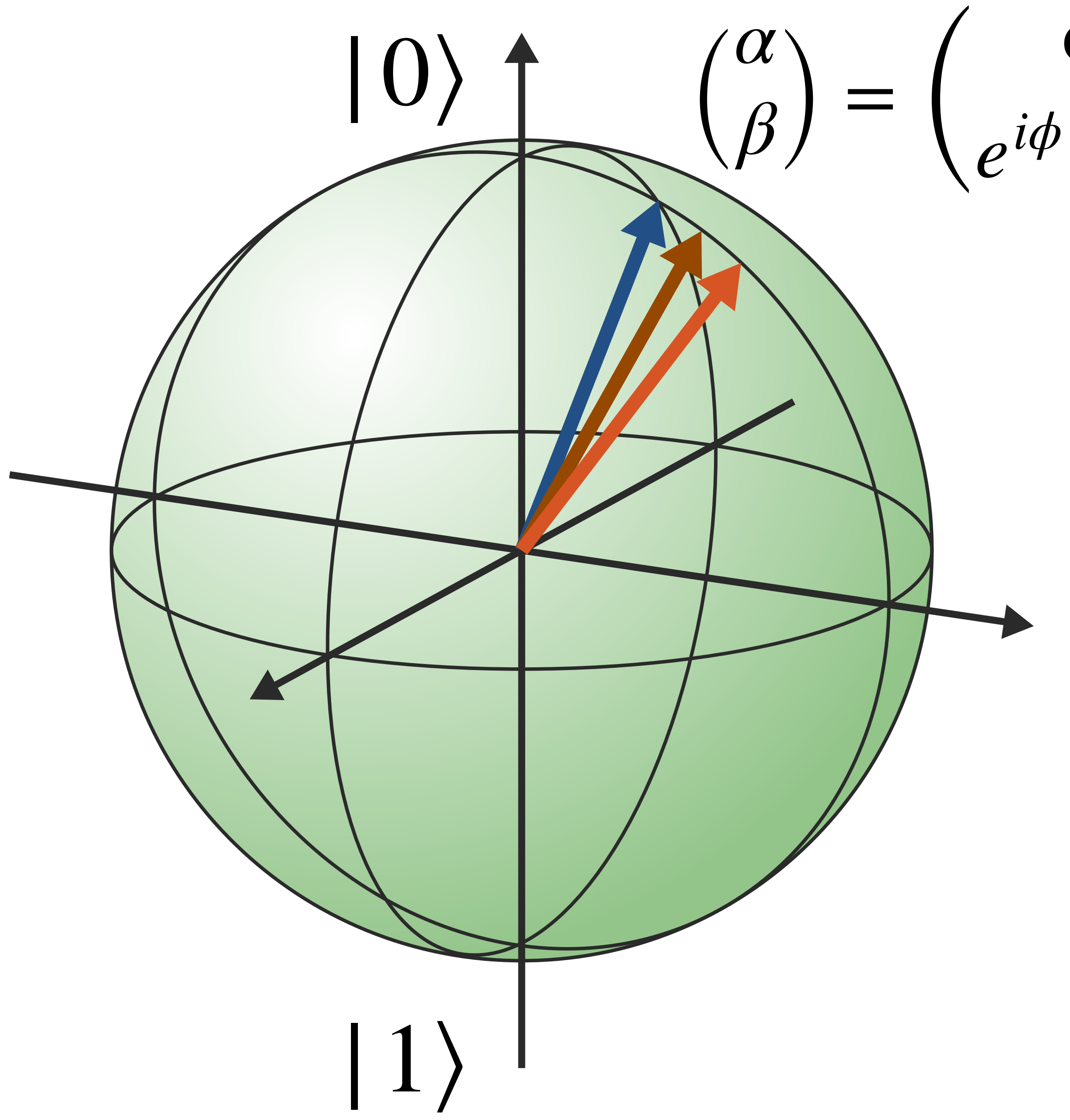

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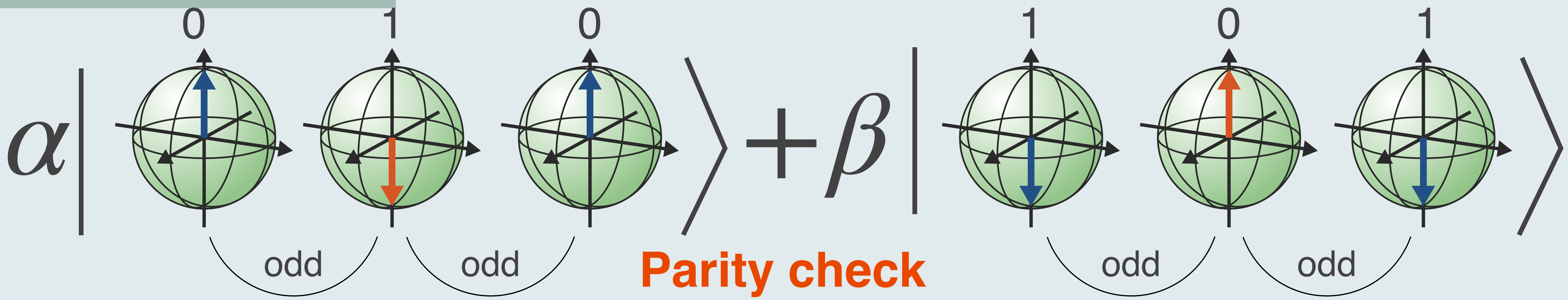

Parity check

Analog noise collapses to a “digital” error.

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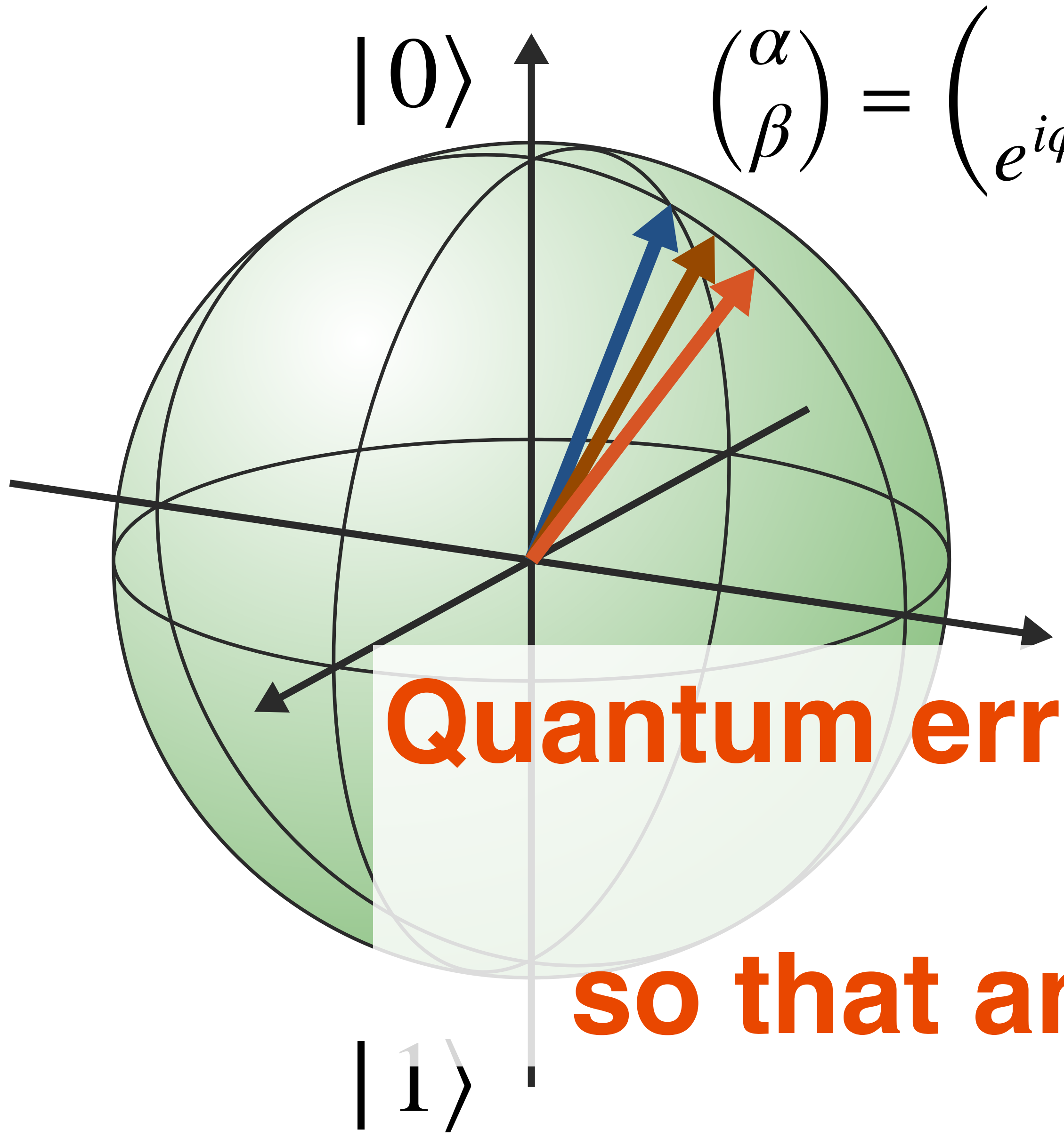

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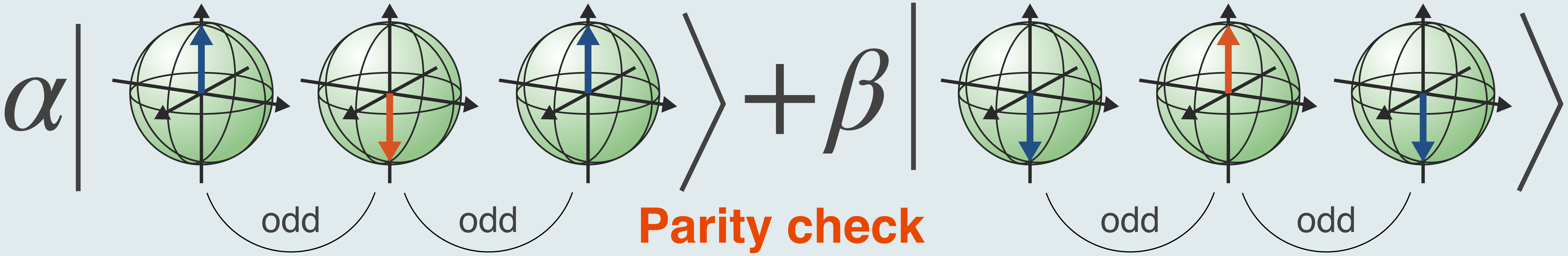
Quantum error correction embeds quantum information into entangled multi-qubit systems so that analog noise can collapses into digital errors.

Quantum Error Correction

Do measurement what "digital" error occurs.



$$\alpha|\ \ \rangle + \beta|\ \ \rangle$$

Parity check

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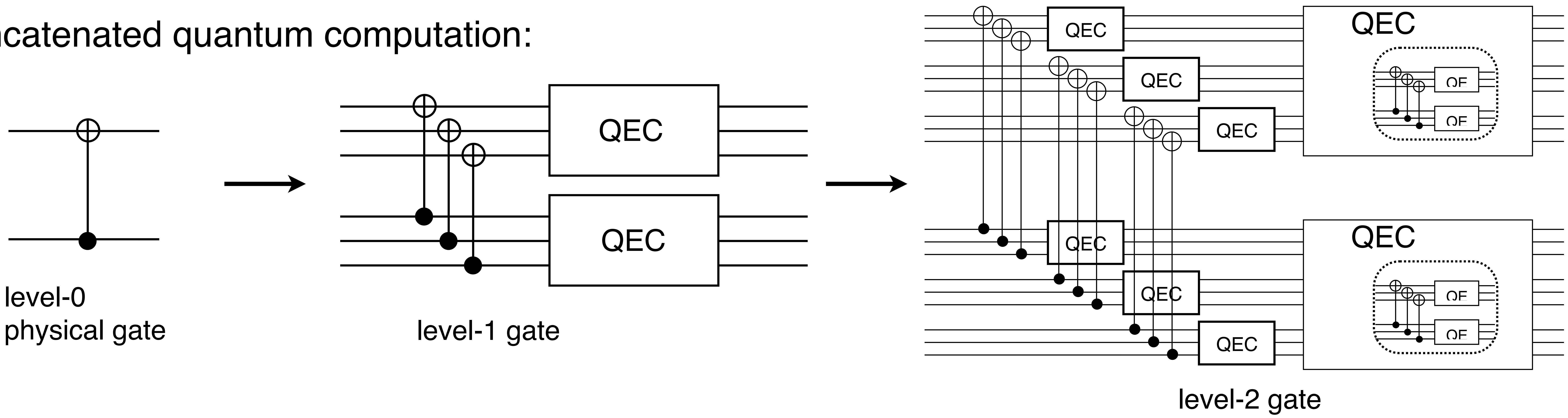
What is fault-tolerance?

Fault-tolerant quantum computing and “Threshold Theorem”

Concatenated quantum computation:



Logical error probability：

$$p \rightarrow Cp^2 \rightarrow C(Cp^2)^2 \rightarrow \cdots \rightarrow (Cp)^{2^l}/C$$

Level-1　Level-2　Level-l

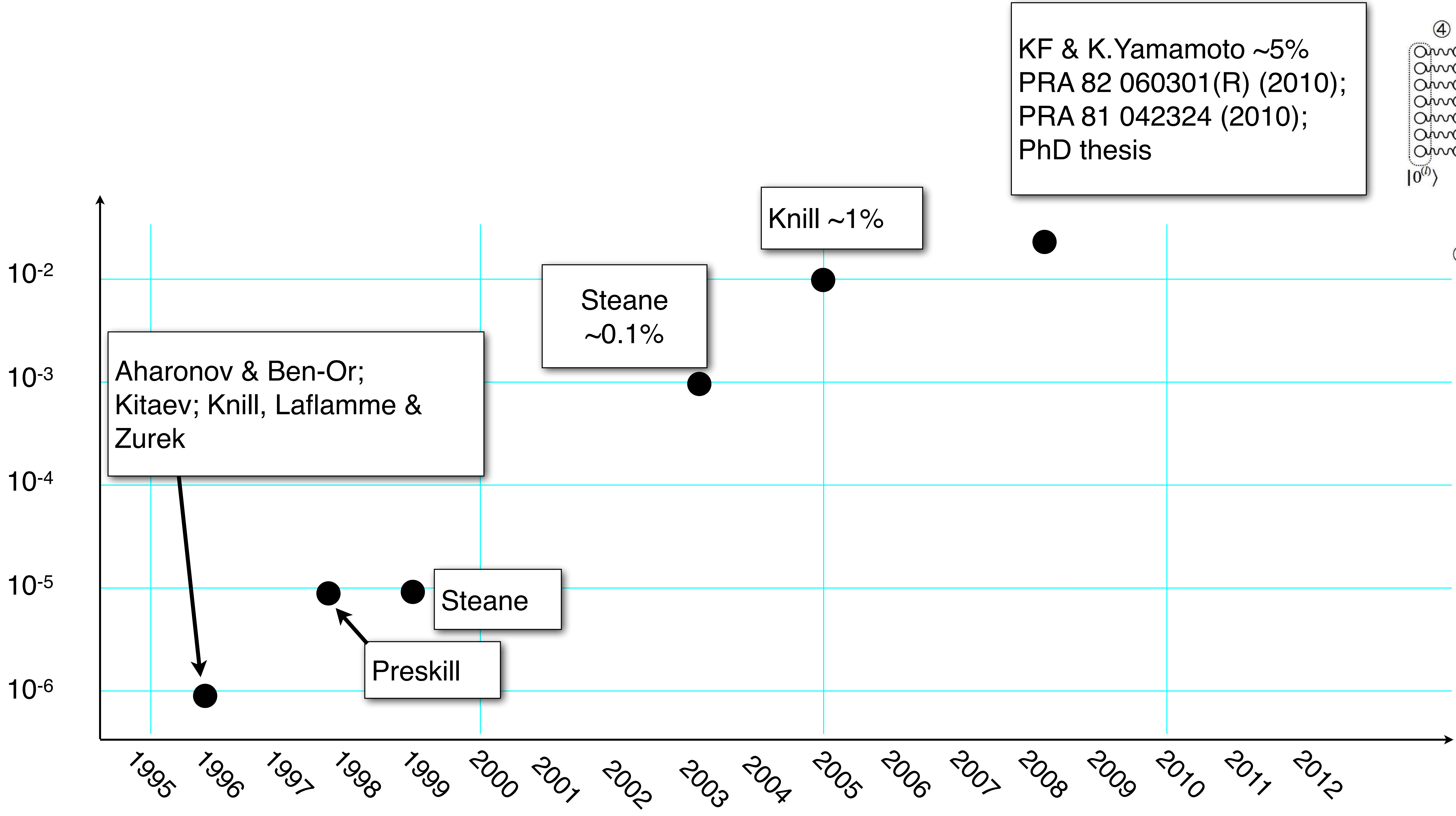
Threshold theorem

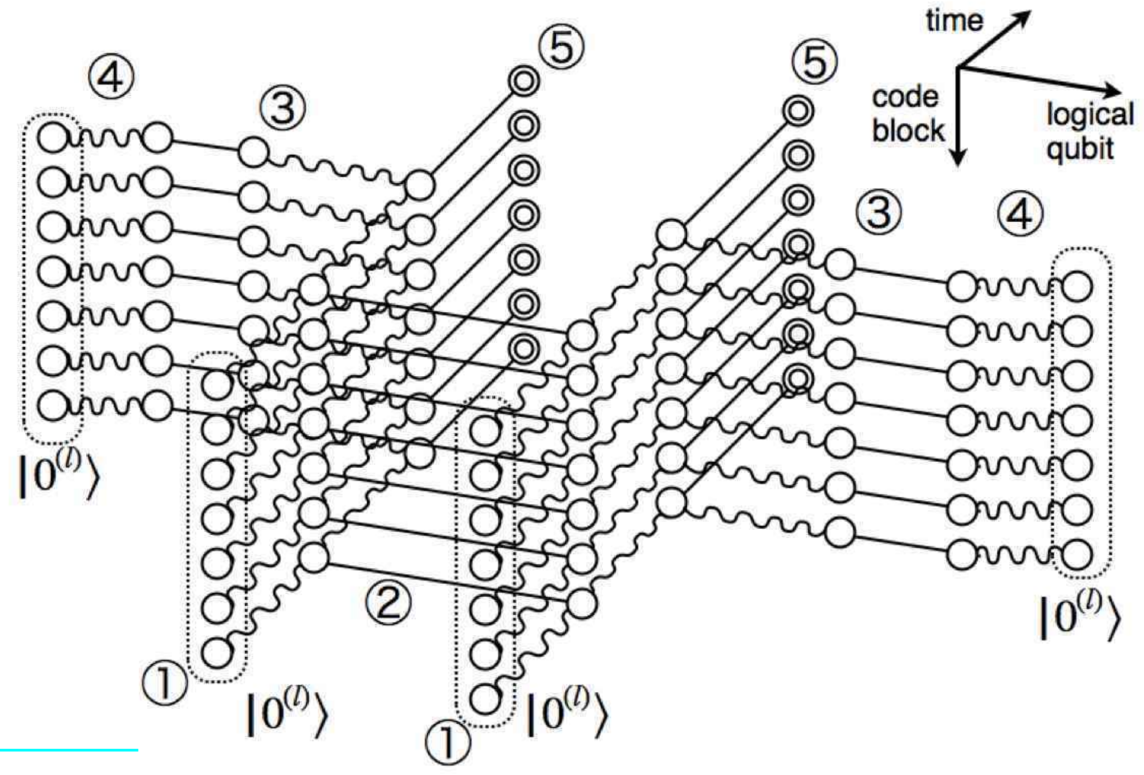
If the error probability is lower than a certain threshold, the logical error probability can be reduced exponentially.

$$p < \underline{1/C (\equiv p_{\mathrm{th}})} \quad \Rightarrow \quad (Cp)^{2^l}/C \rightarrow 0$$

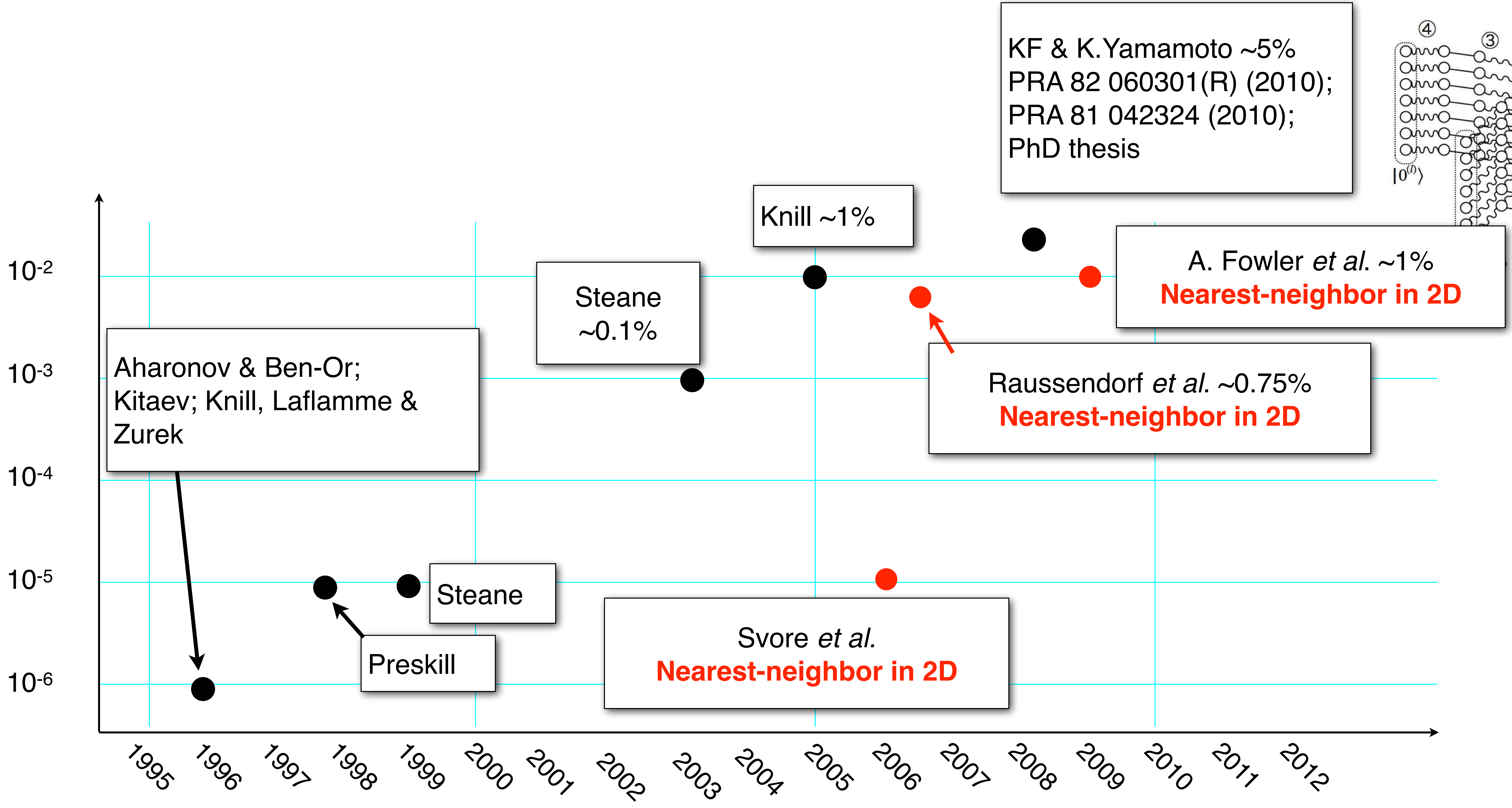
Threshold value

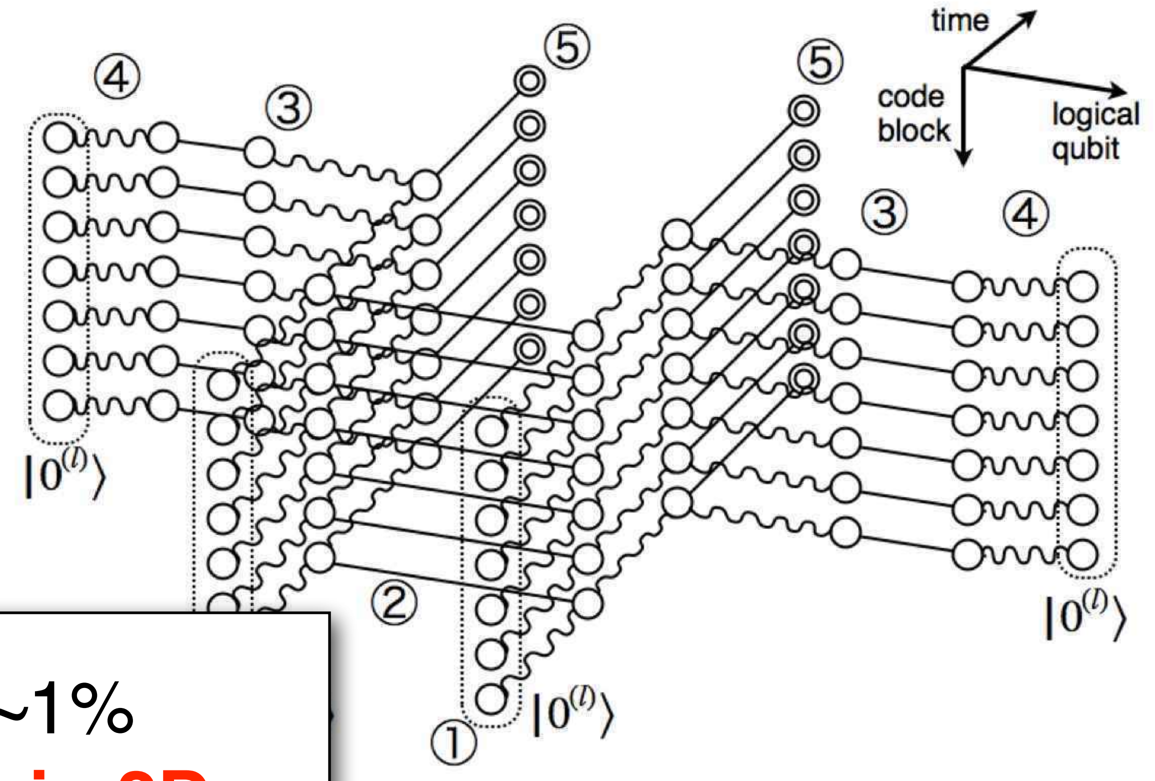
History of Threshold Values





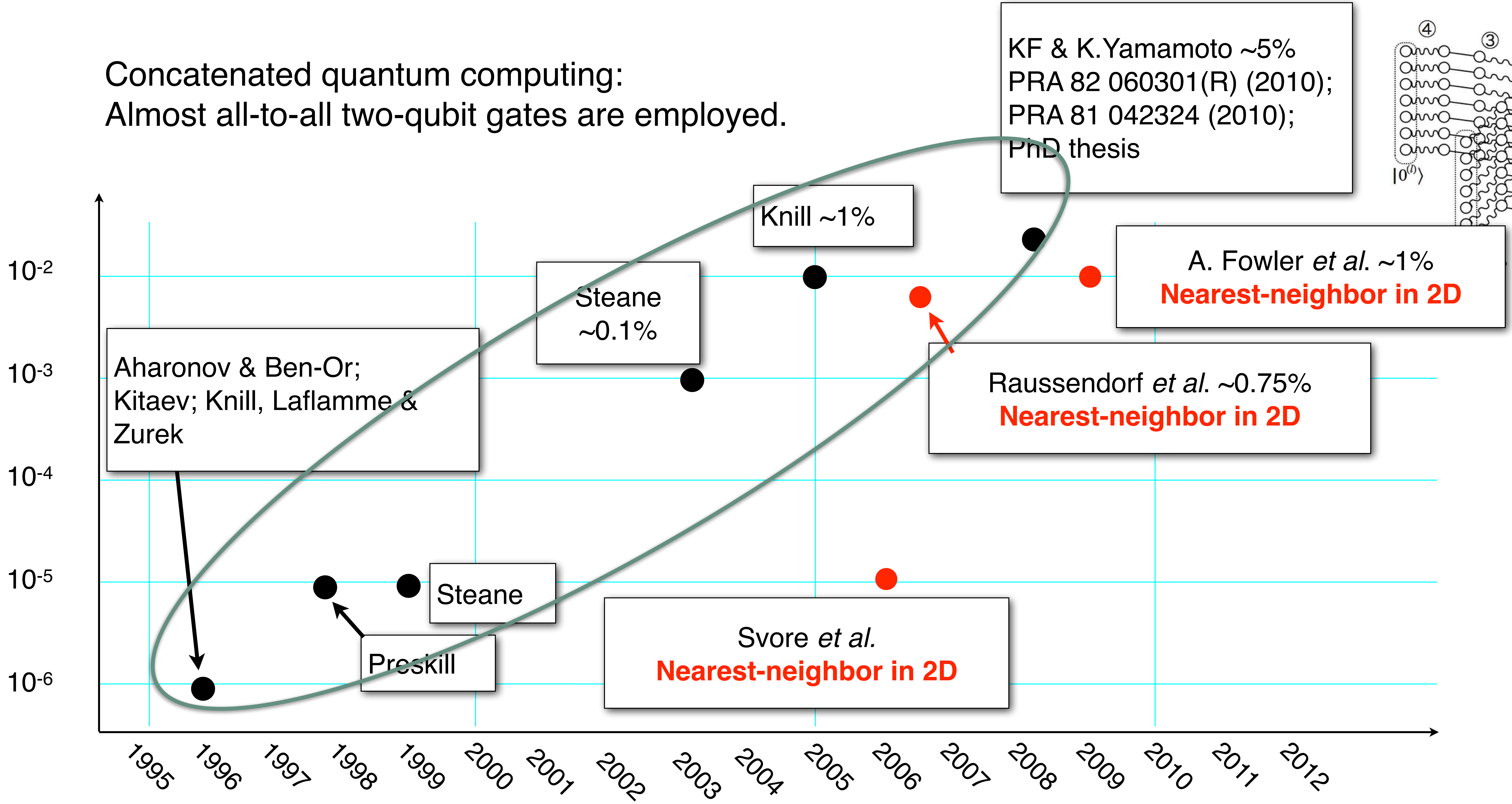
History of Threshold Values

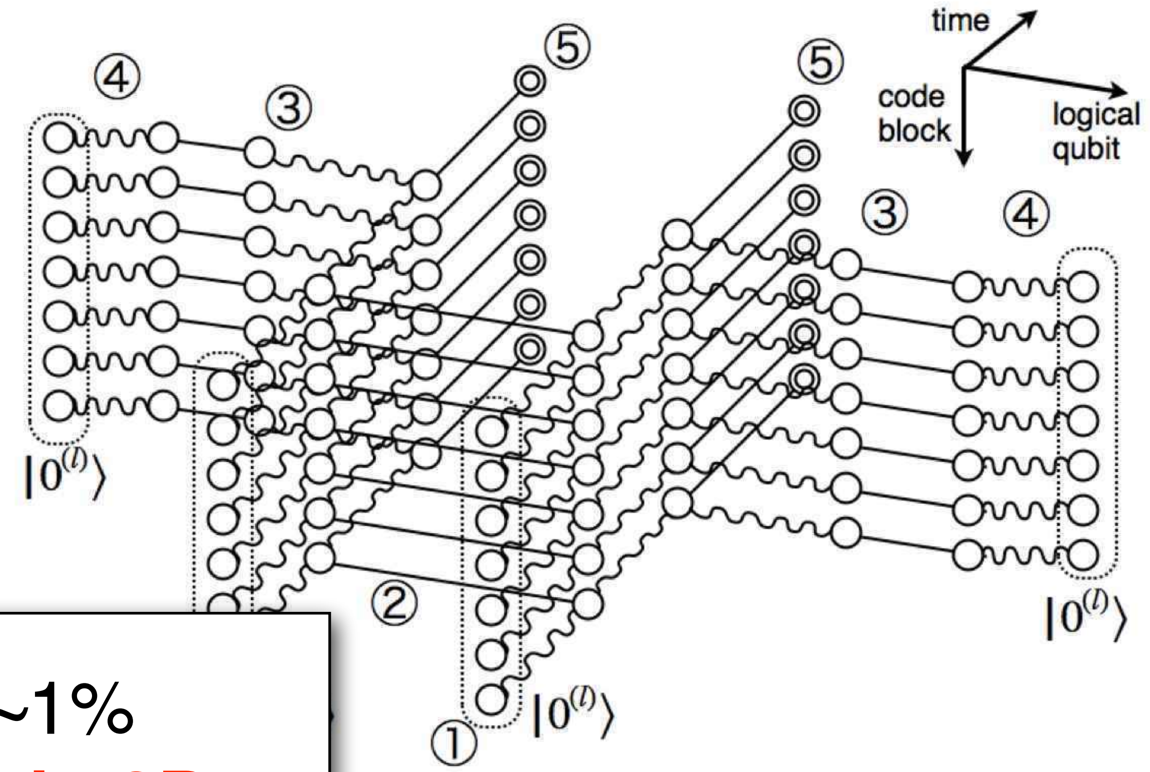




History of Threshold Values

Concatenated quantum computing:
Almost all-to-all two-qubit gates are employed.





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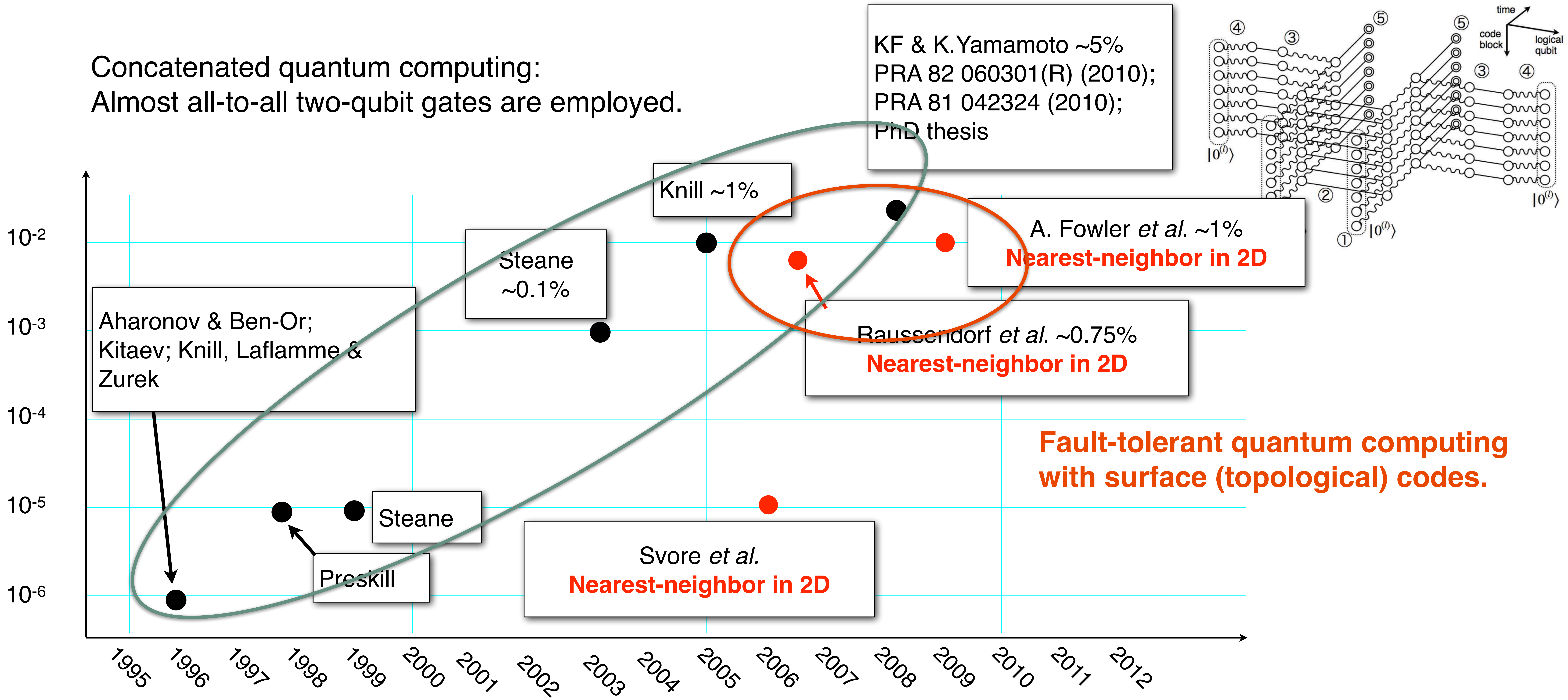
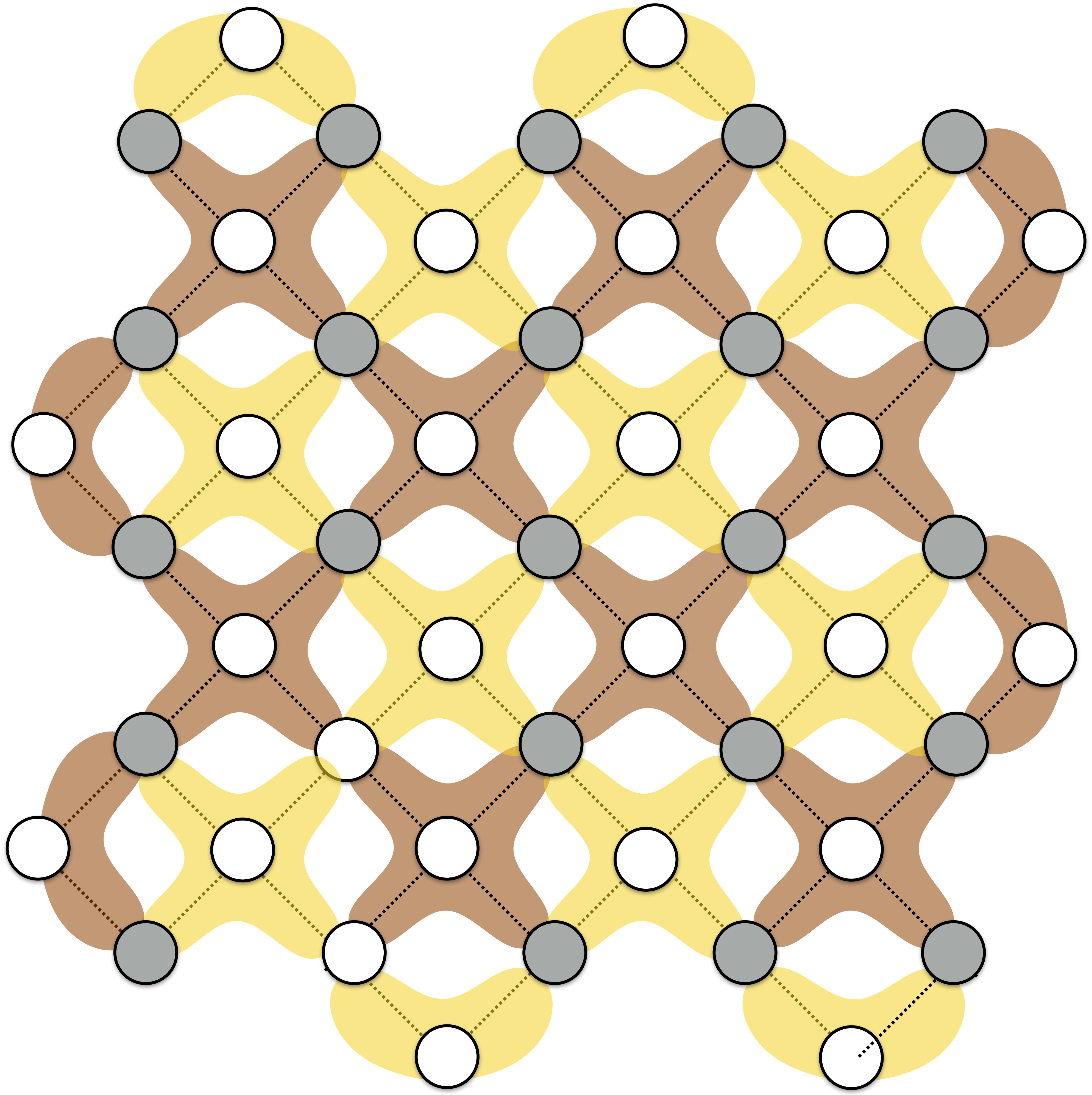
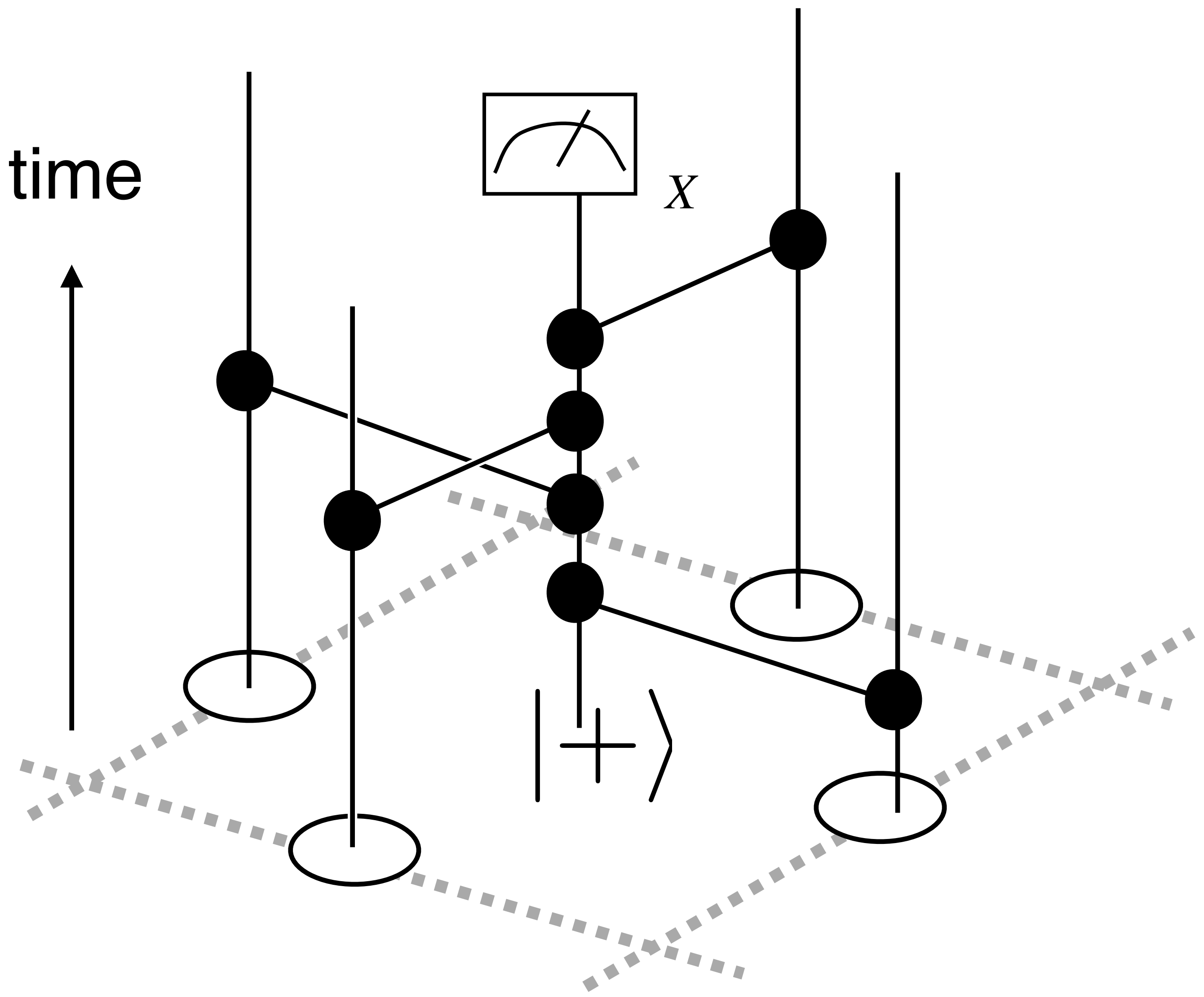


The surface codes



Data qubit.

Syndrome qubit to check parity of neighboring qubits.



Syndrome measurement circuit.

Threshold value ~ 1% (fidelity 99%)

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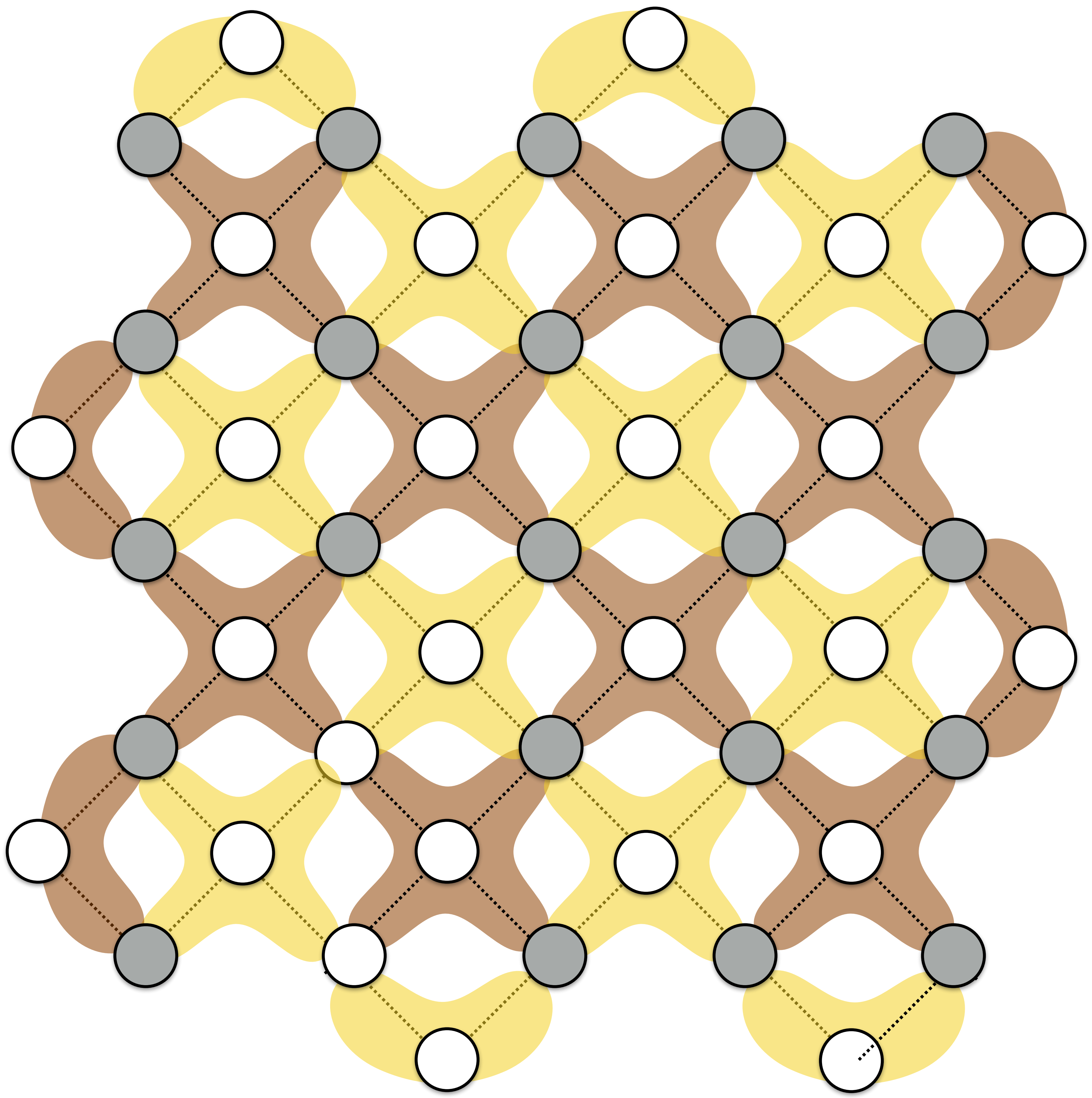

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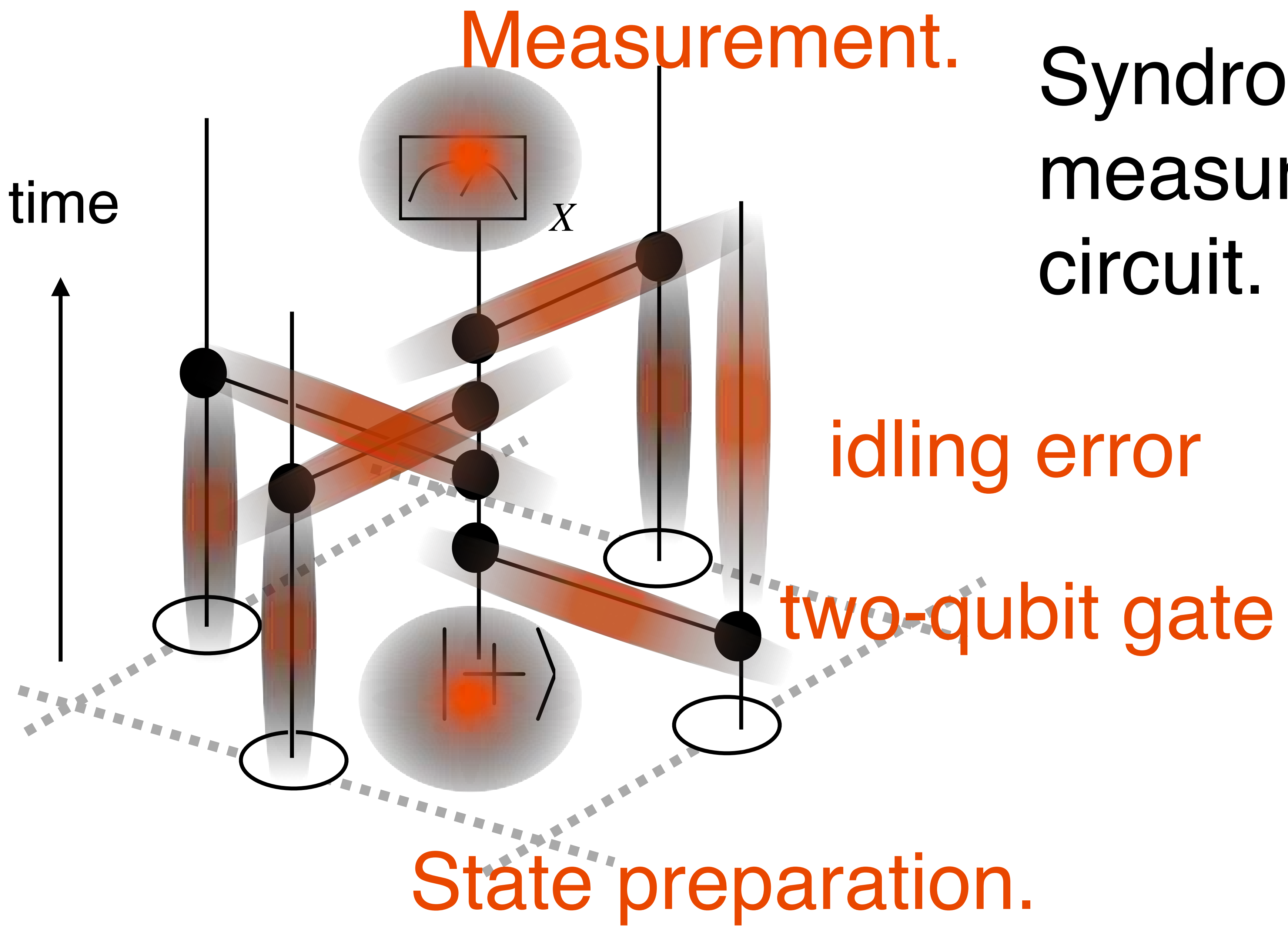

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Google entered quantum computing race in 2014

UCSB J. Martinis's group:

*"Superconducting quantum circuits **at the surface code threshold for fault tolerance**"*

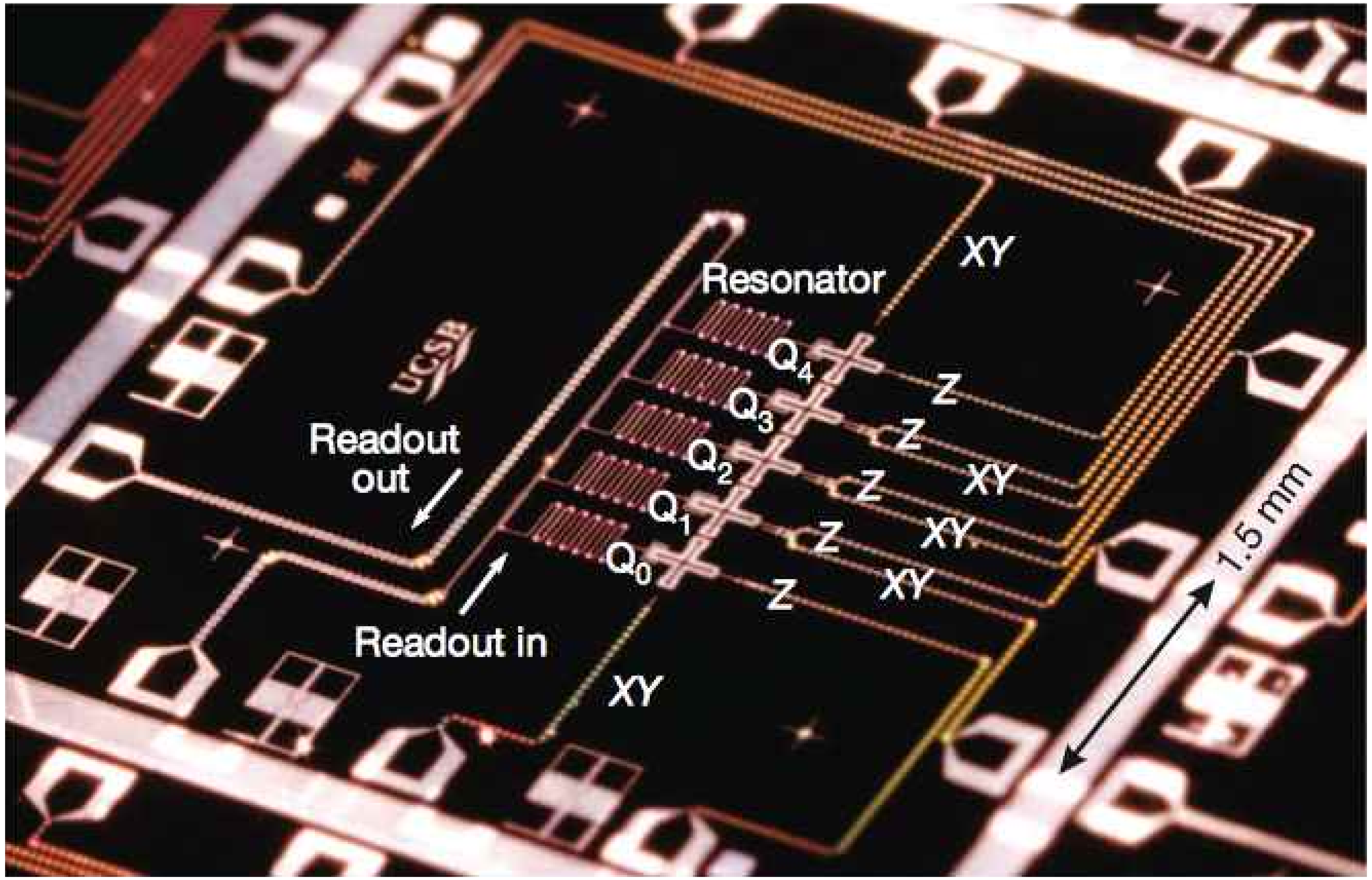
*Barends et al., Nature **508**, 500 (2014)*

[ultra high fidelity]

single-qubit gate : 99.92%

two-qubit gate : 99.4%

measurement : 99%



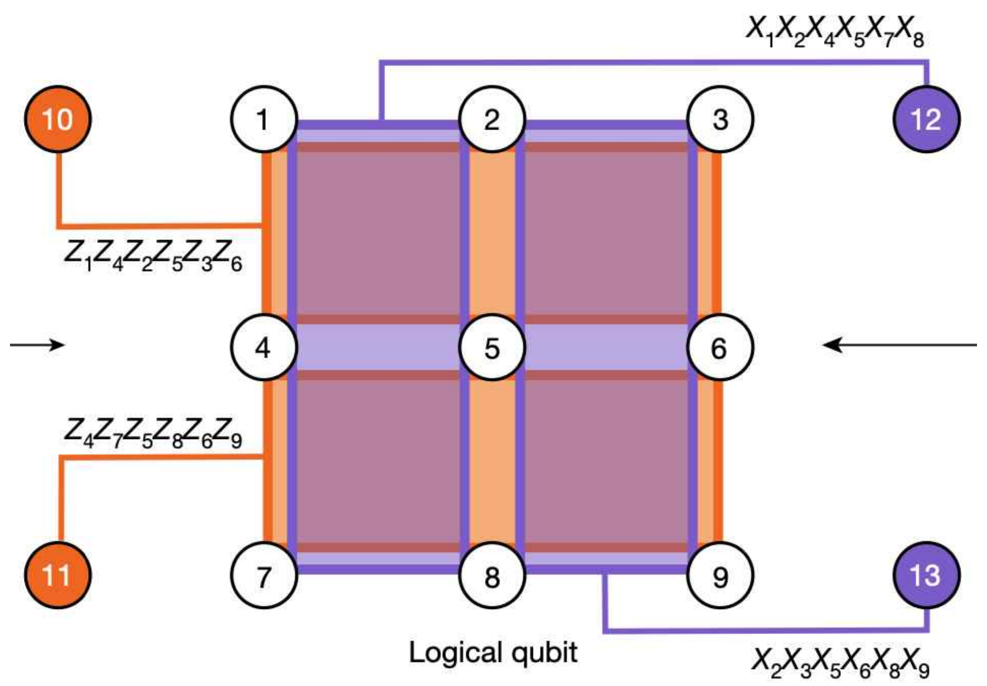
Quantum computer quest:

Nature News feature

Nature **516**, 24-26 (2014)

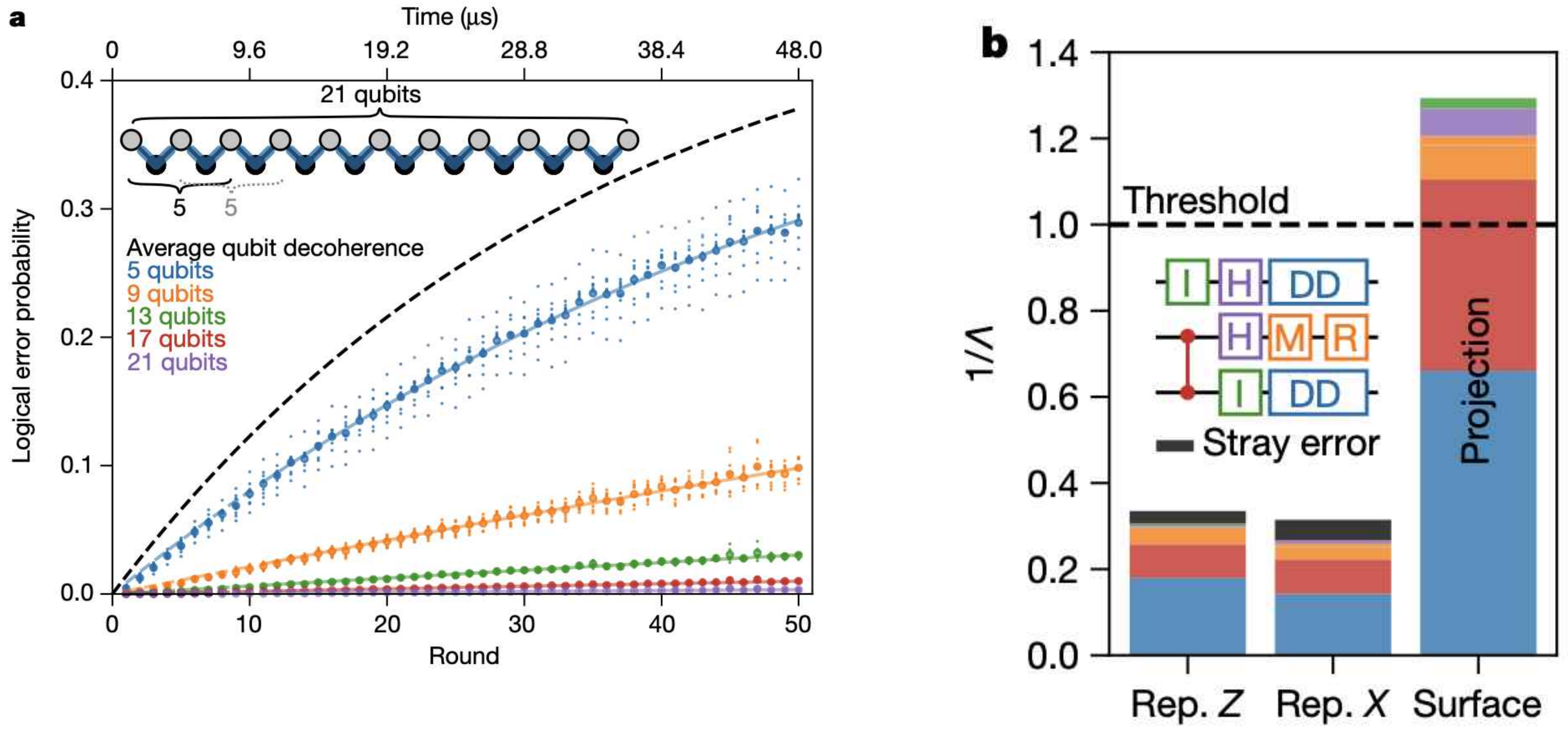
2023: First year of Practical Quantum Error Correction?

Syndrome measurement for Bacon-Shor code



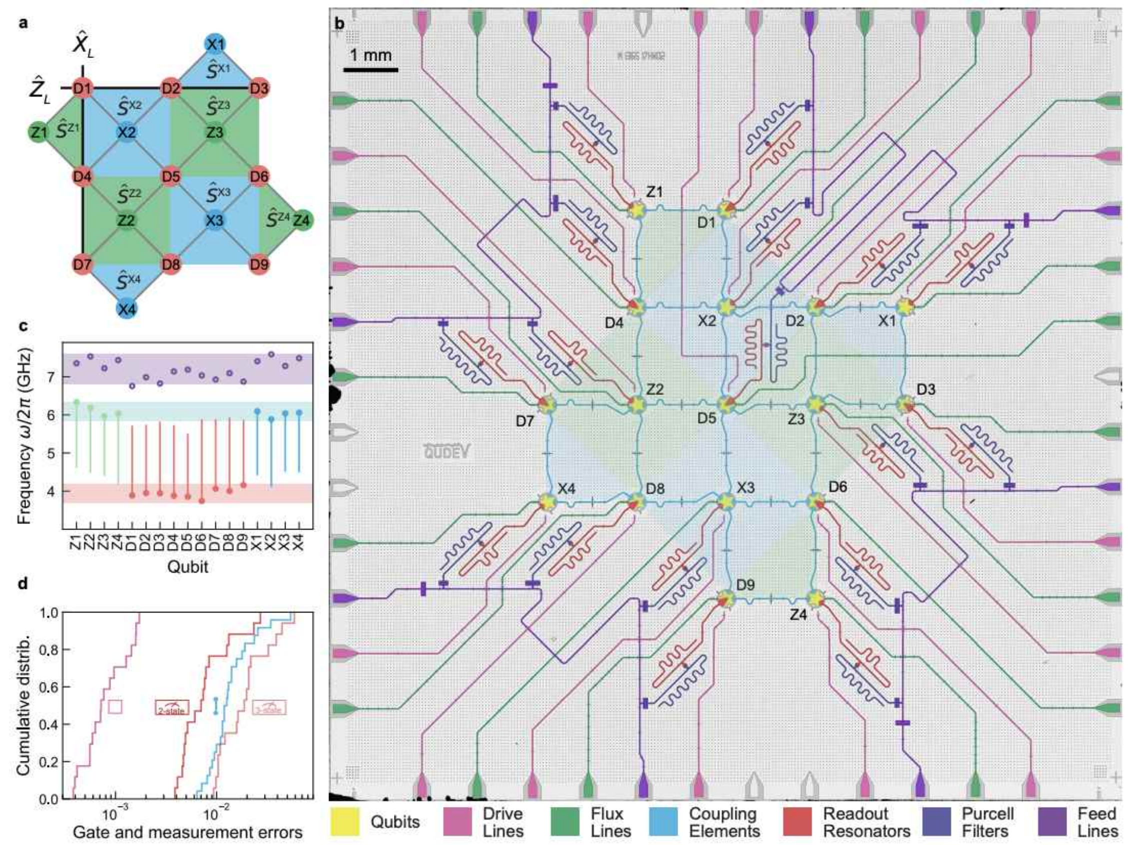
L. Egan et al. "*Fault-tolerant control of an error-corrected qubit.*" Nature **598**, 281 (2021).

21-qubit one-dimensional repetition code



AI, Google Quantum. *"Exponential suppression of bit or phase errors with cyclic error correction."* Nature **595**, 383 (2021).

17-qubit surface code



S. Krinner et al., *"Realizing Repeated Quantum Error Correction in a Distance-Three Surface Code",* Nature **605**, 669 (2022)

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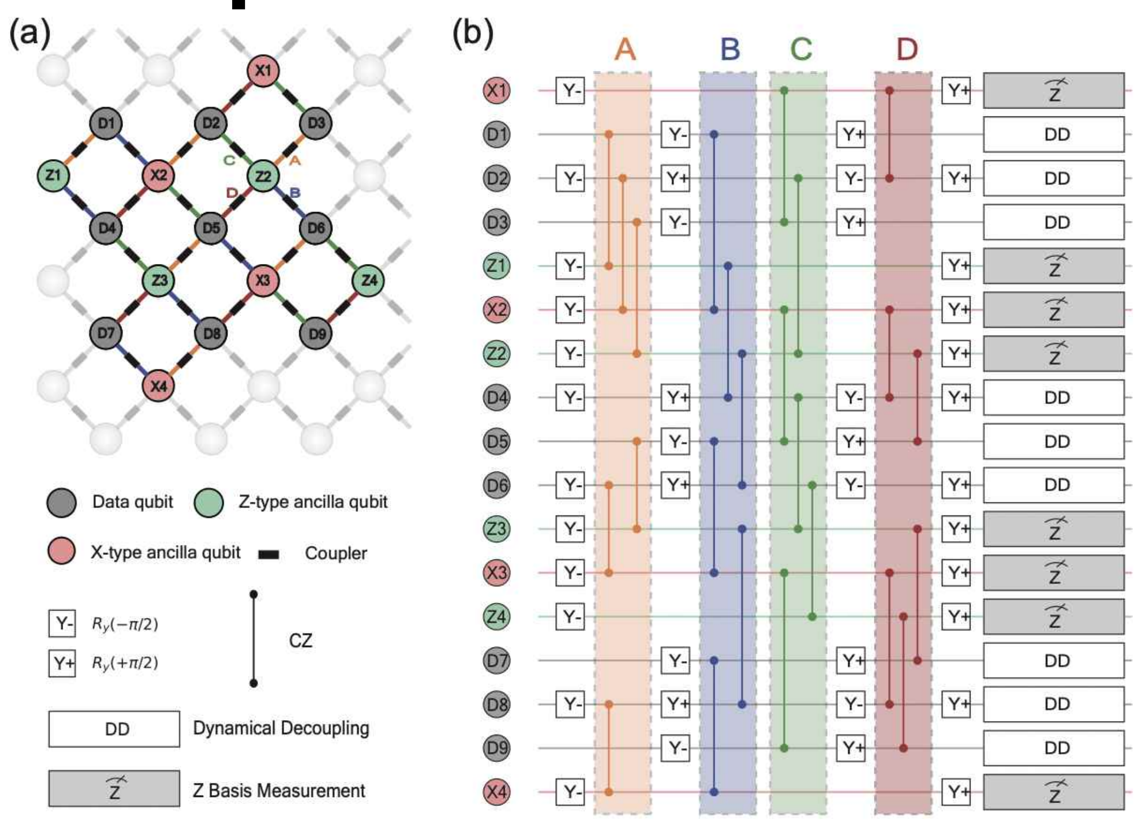
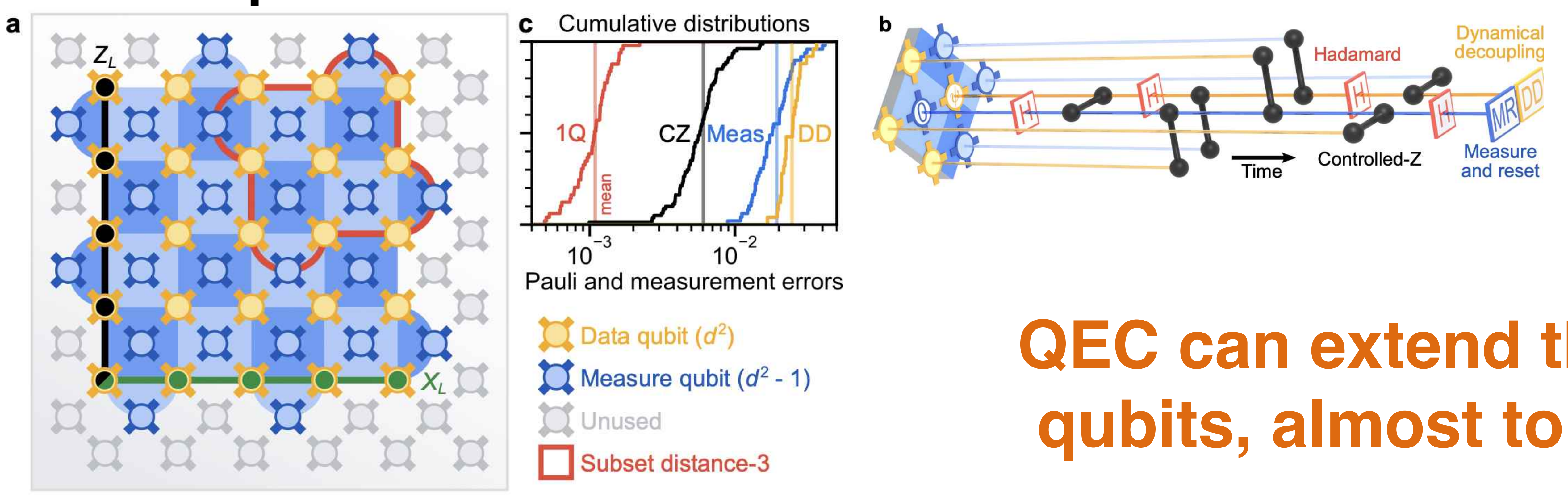


Y. Zhao et al. "*Realization of an error-correcting surface code with superconducting qubits.*" Physical Review Letters **129,** 030501 (2022).

49-qubit surface code



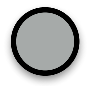
AI, Google Quantum. *"Suppressing quantum errors by scaling a surface code logical qubit." Nature*, **614**, 676 (2023).

QEC can extend the lifetime of qubits, almost to break-even.

Fault-tolerant quantum computing: whole picture

R. Raussendorf and J. Harrington. "*Fault-tolerant quantum computation with high threshold in two dimensions.*" PRL (2007).
A. G. Fowler *et al.* *"Surface codes: Towards practical large-scale quantum computation."* PRA (2012).

qubit



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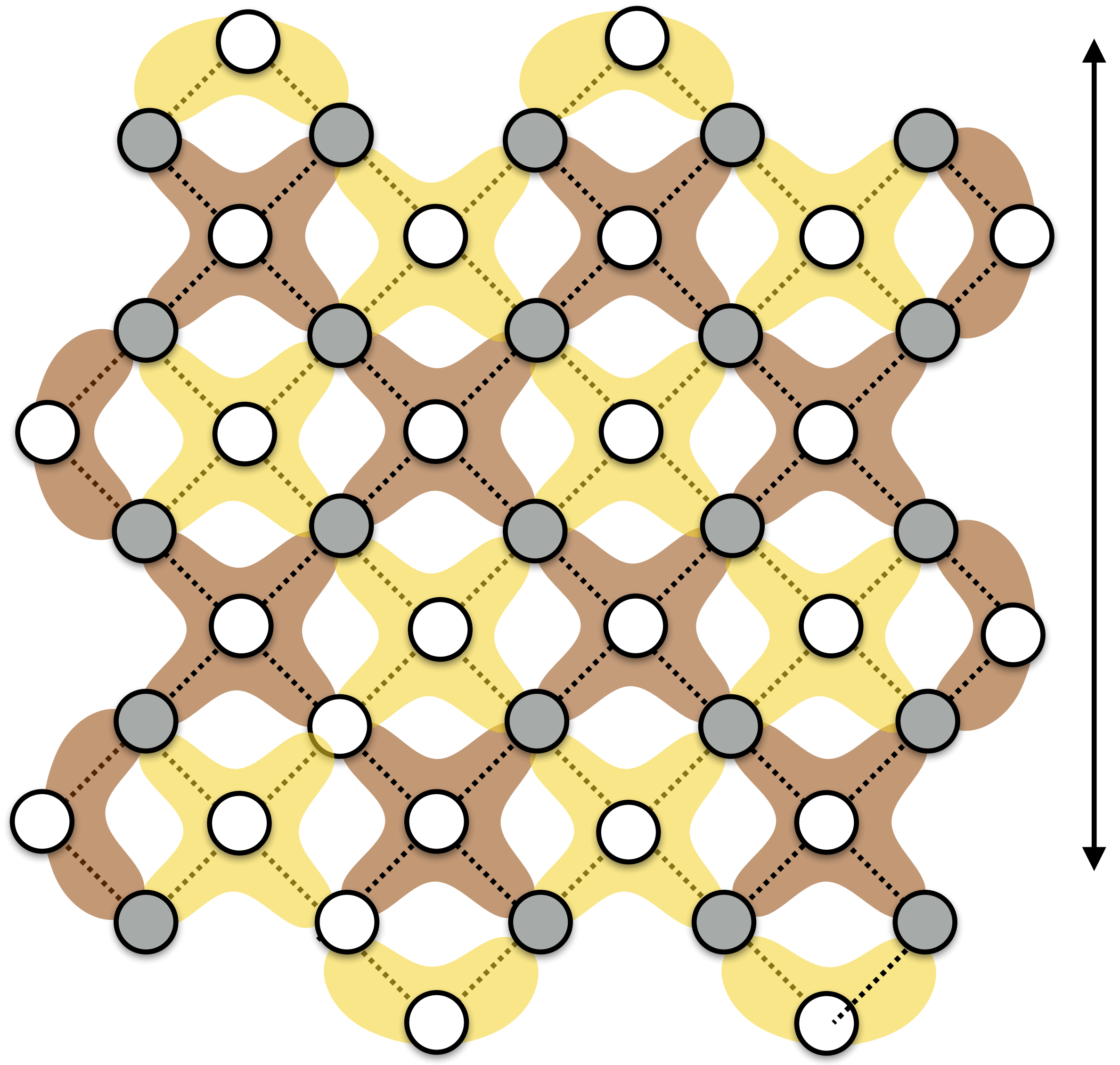

code distance d

logical qubit: $2d^2 - 1$ physical qubits

logical error rate : $p_L = C \left(\dfrac{p}{p_{\text{th}}} \right)^{(d+1)/2}$

ex) $p = 10^{-3}, p_{\text{th}} = 10^{-2}, d = 19, p_L \sim 10^{-10}$

→721 physical qubits per logical qubit

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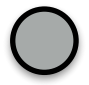

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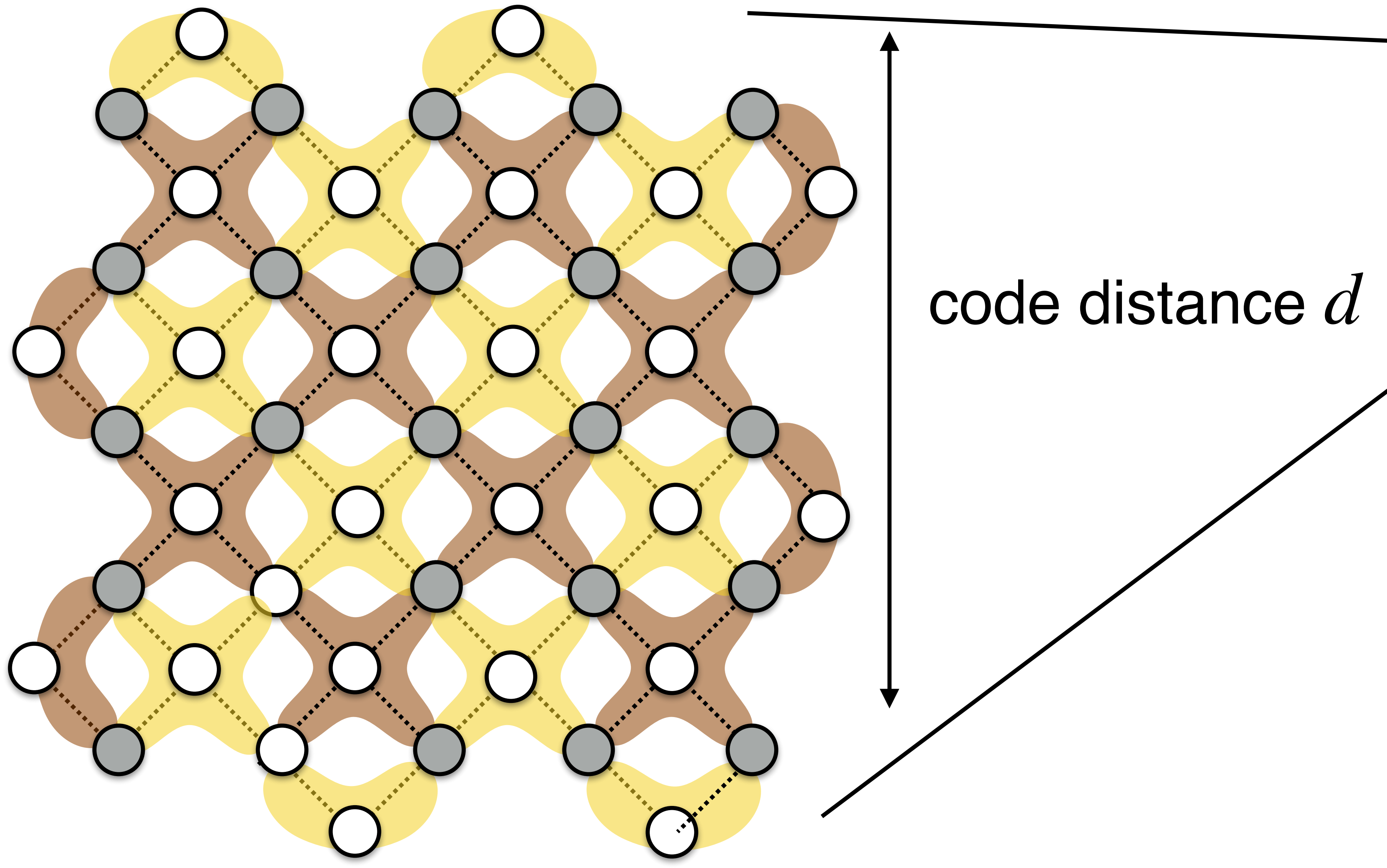

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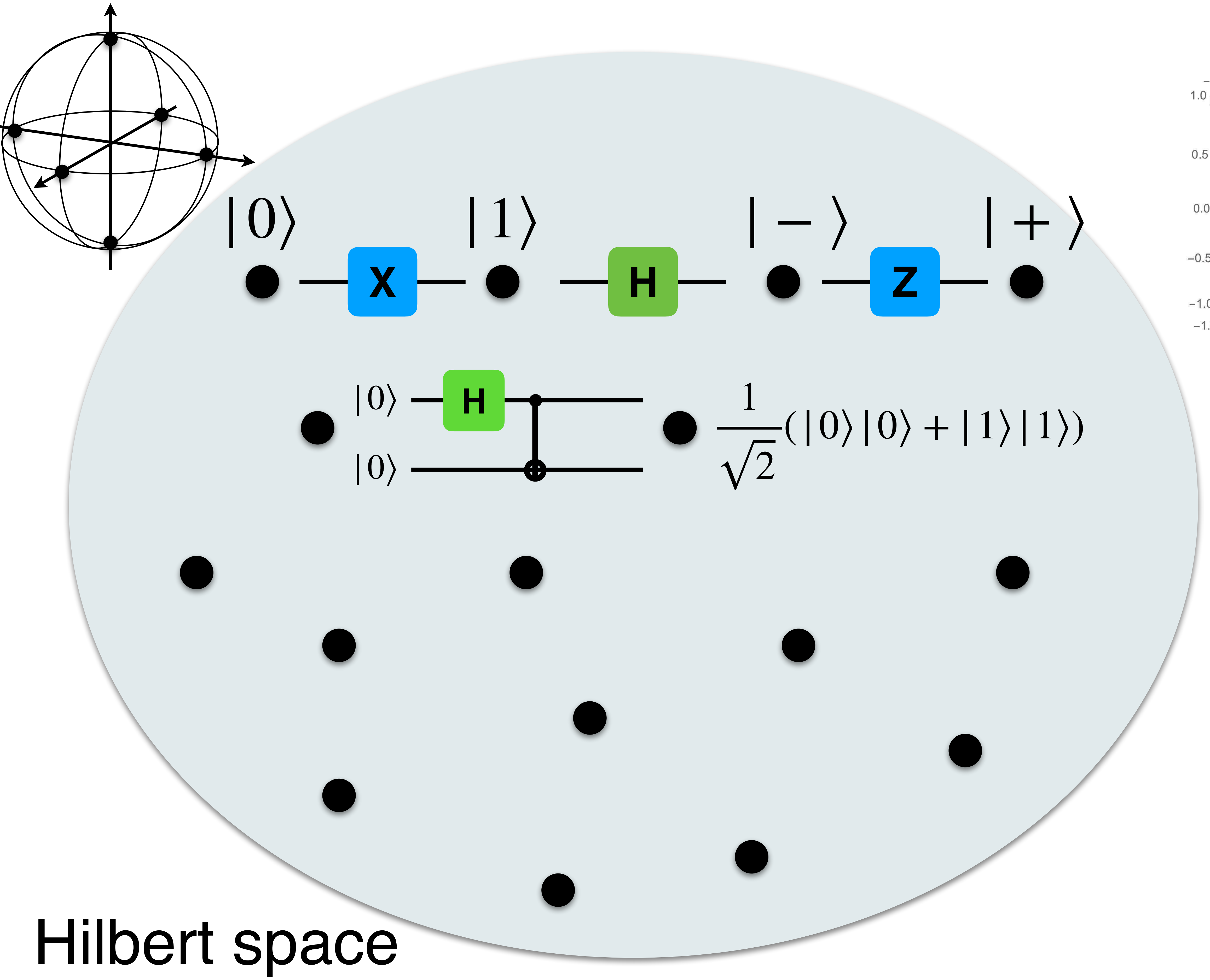
D. Litinski Quantum (2019); M. Beverland, K. Vadym, and E. Schoute, PRX Quantum (2022).

Quantum algorithm using 100-1000 logical qubits

Total amount of physical qubits → 10^6

Clifford gates and Non-Clifford gates

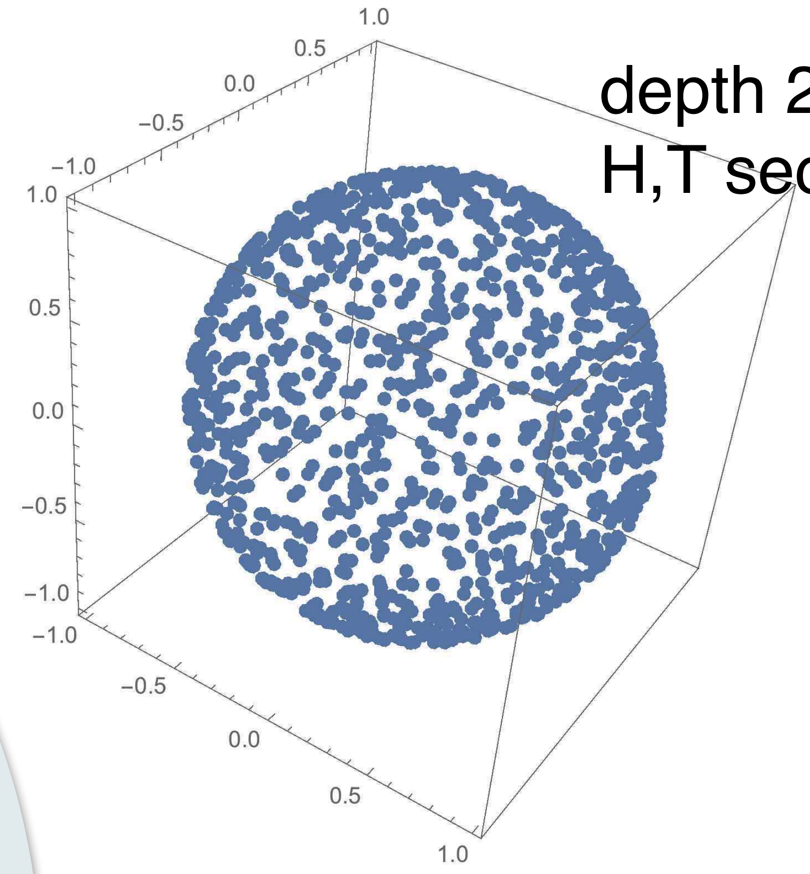
Clifford gates: X,Y, Z, H, S, CNOT, CZ



Hilbert space

The states generated by Clifford gates are rather limited and can be simulated efficiently (Gottesman-Knill's theorem).

Non-Clifford gates: T, Toffoli gates



depth 20
H,T sequence

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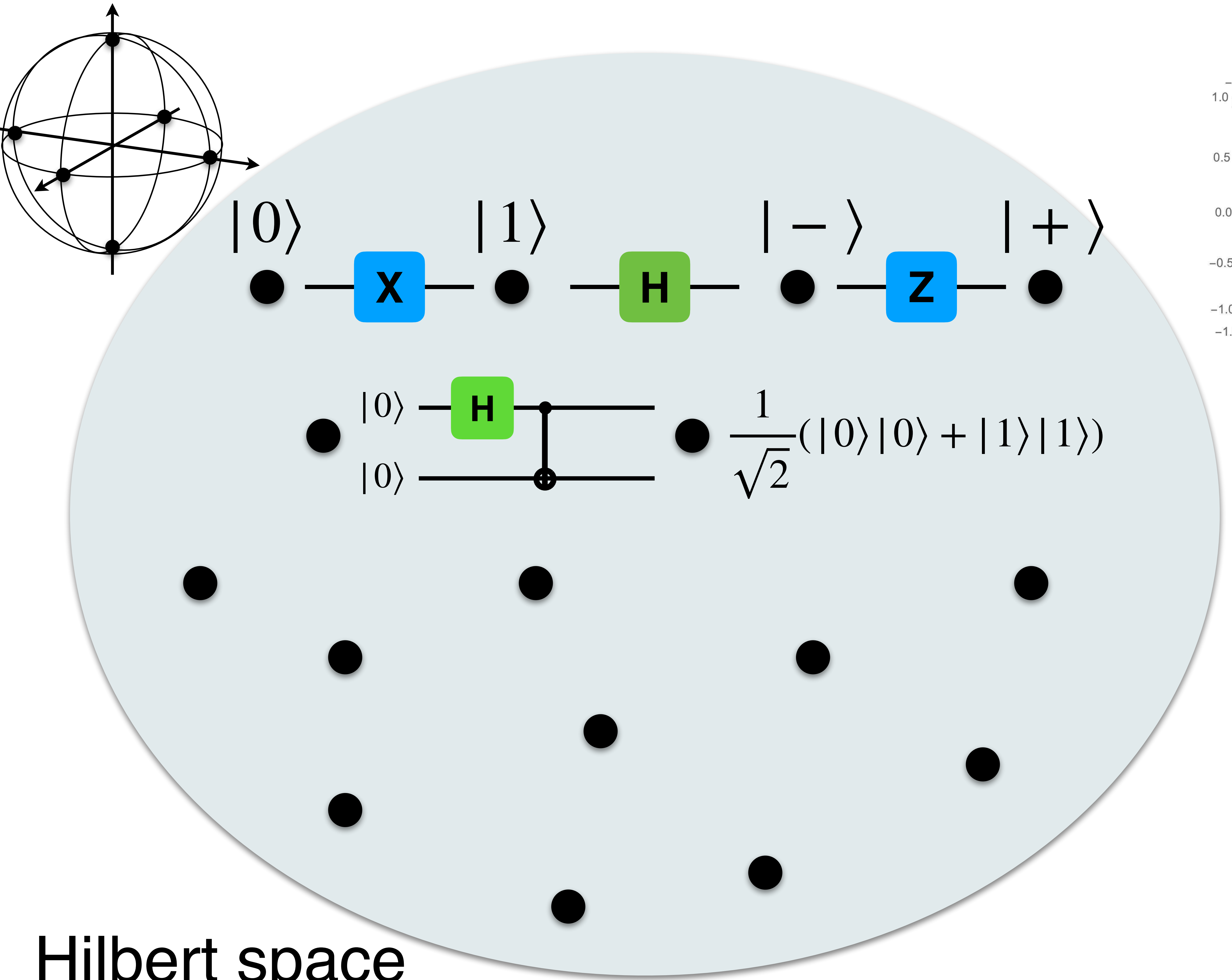

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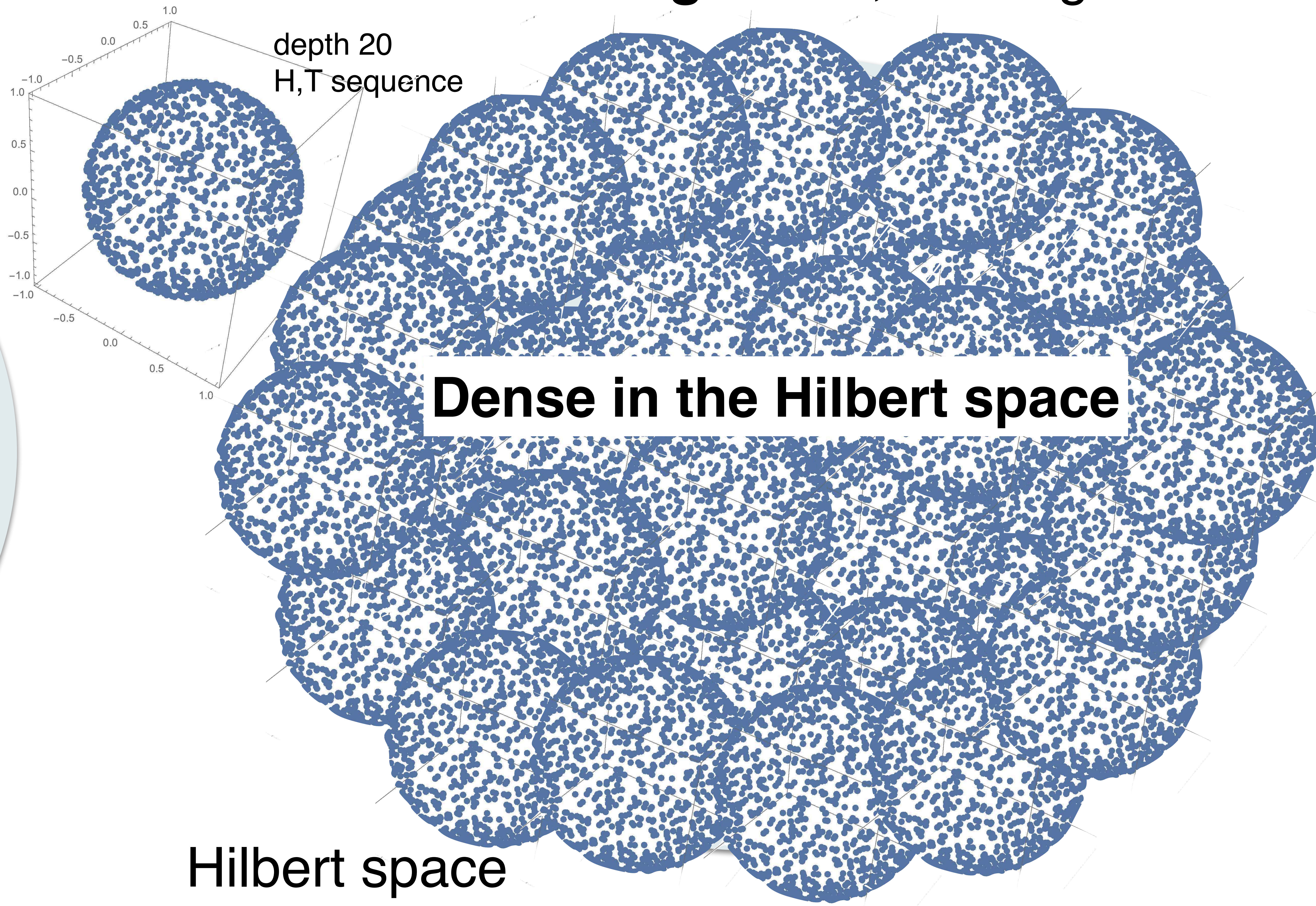

Hilbert space

Clifford + non-Clifford gates
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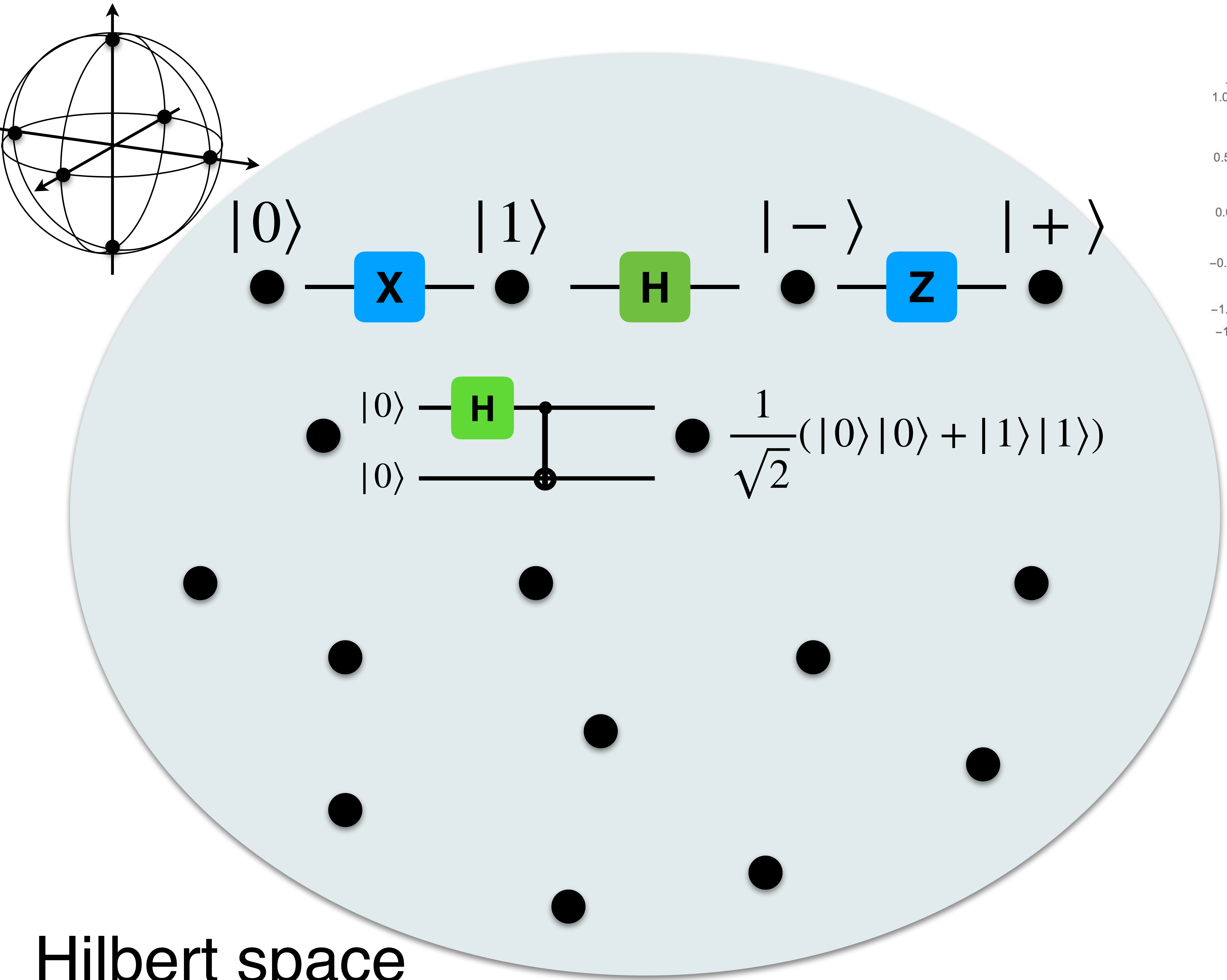

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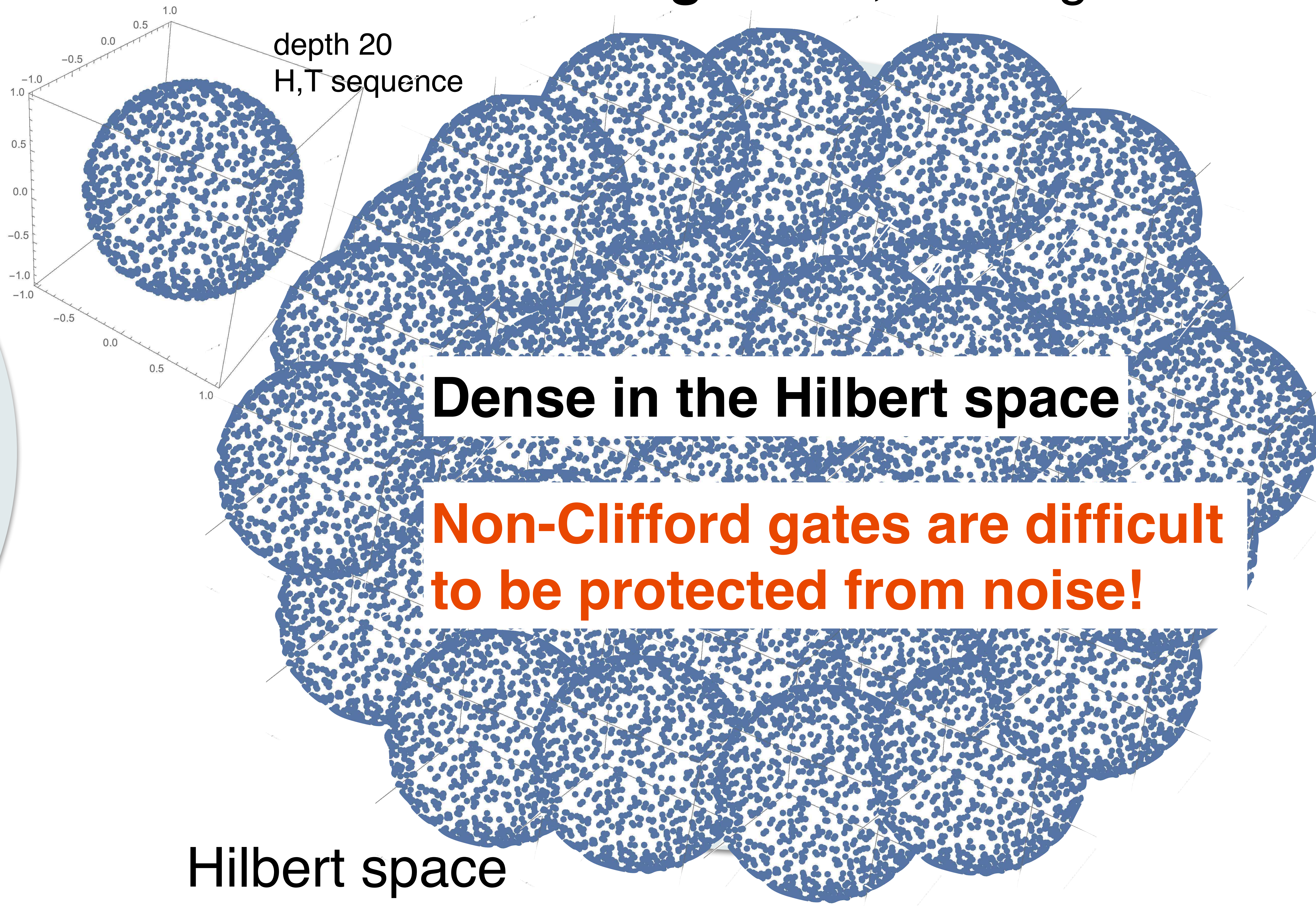

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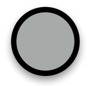
qubit



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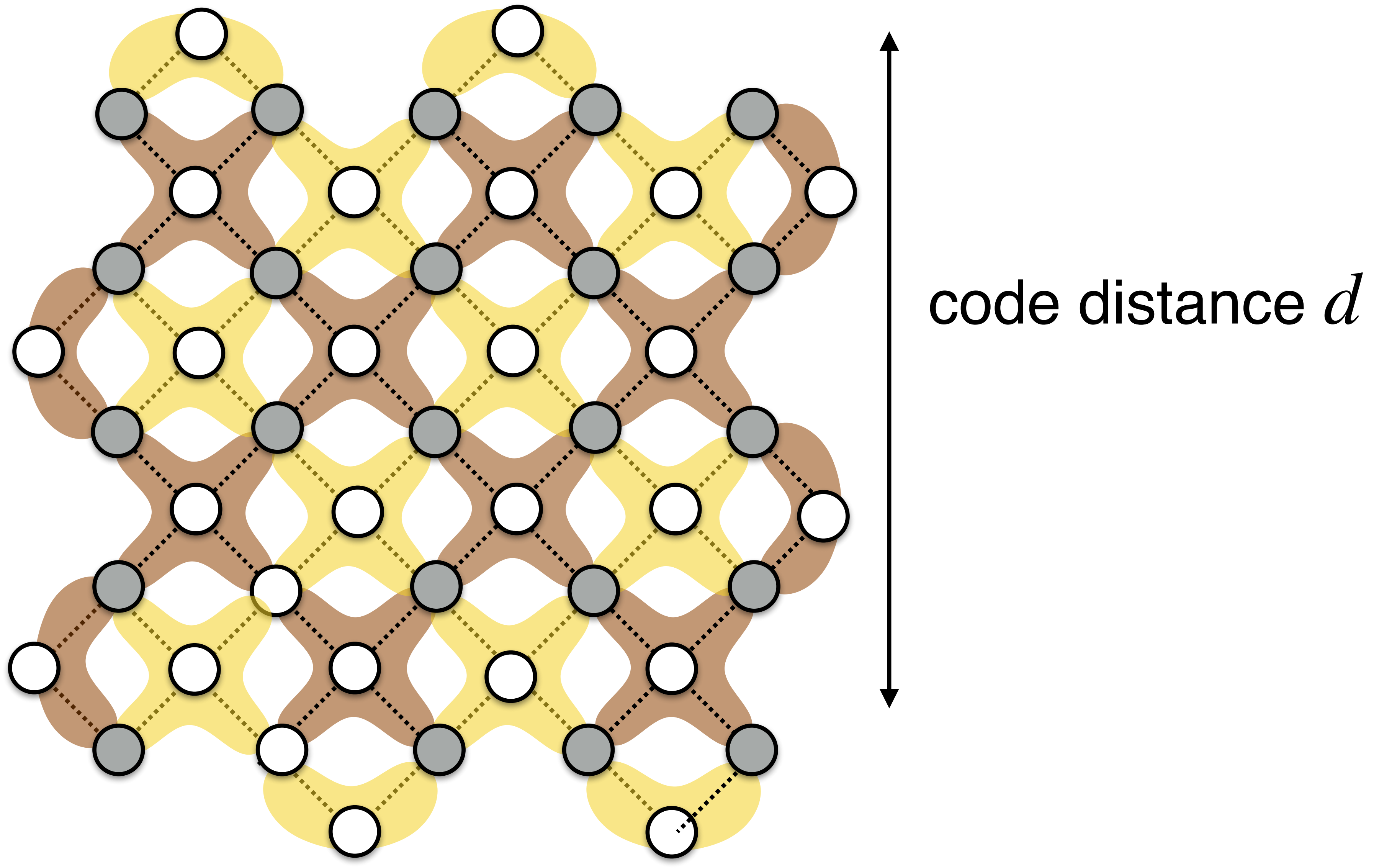

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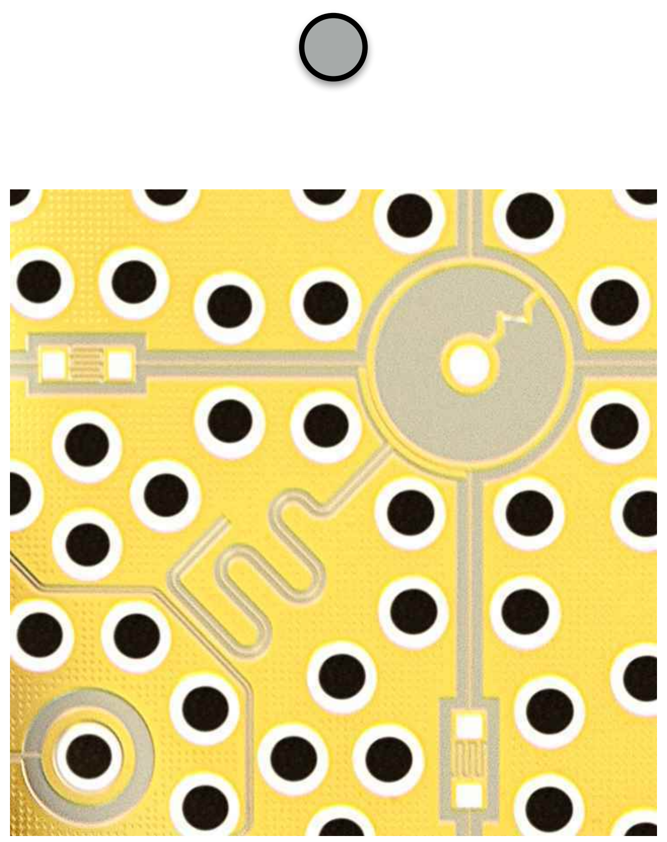
qubit



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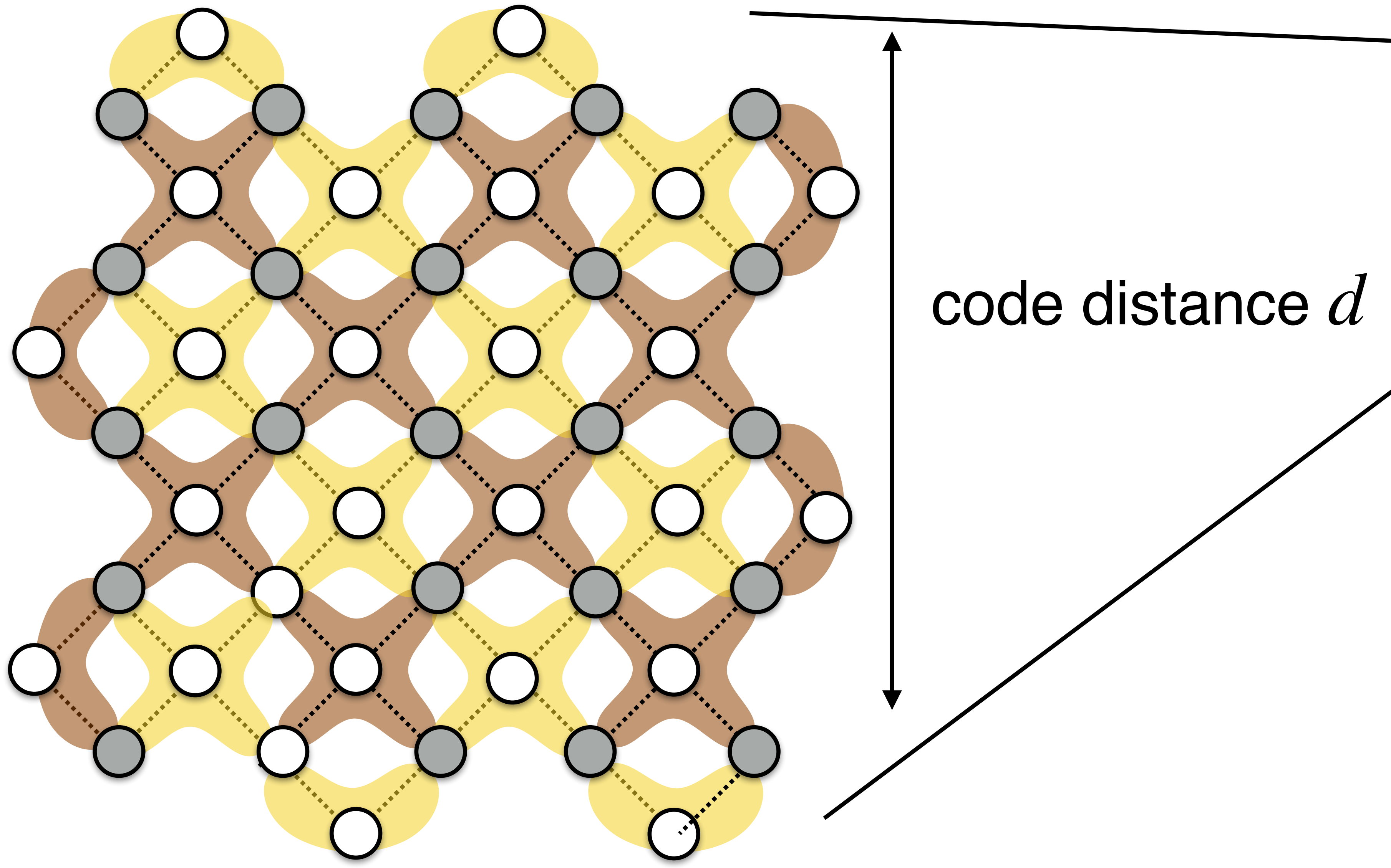

D. Litinski Quantum (2019); M. Beverland, K. Vadym, and E. Schoute, PRX Quantum (2022).

logical qubit: $2d^2 - 1$ physical qubits

logical error rate : $p_L = C\left(\frac{p}{p_{\rm th}}\right)^{(d+1)/2}$

ex) $p = 10^{-3}, p_{\rm th} = 10^{-2}, d = 19, p_L \sim 10^{-10}$

→721 physical qubits per logical qubit

Quantum algorithm using 100-1000 logical qubits

Total amount of physical qubits → 10^6

Fault-tolerant quantum computing: whole picture

R. Raussendorf and J. Harrington. "*Fault-tolerant quantum computation with high threshold in two dimensions.*" PRL (2007).
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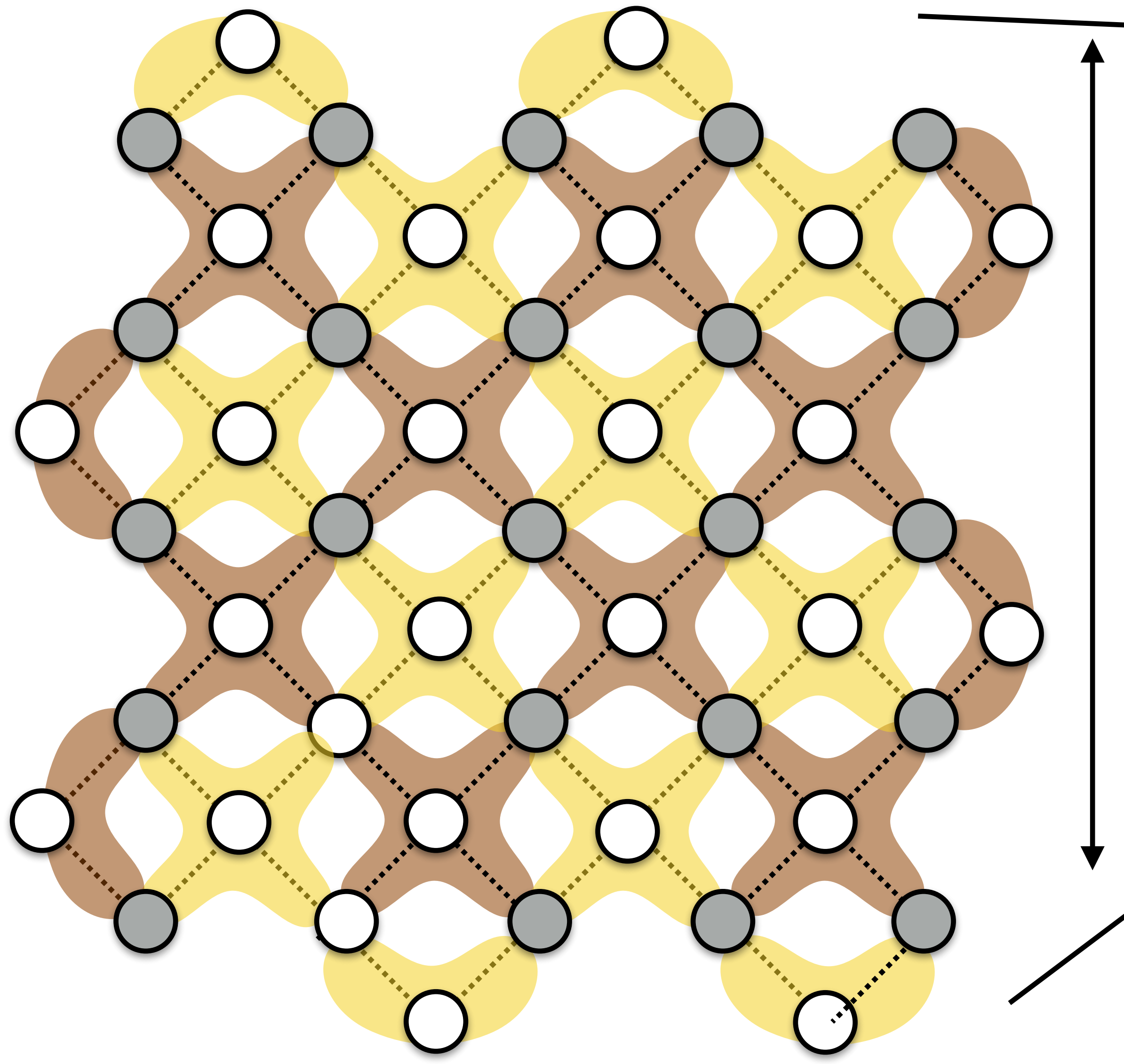

code distance d

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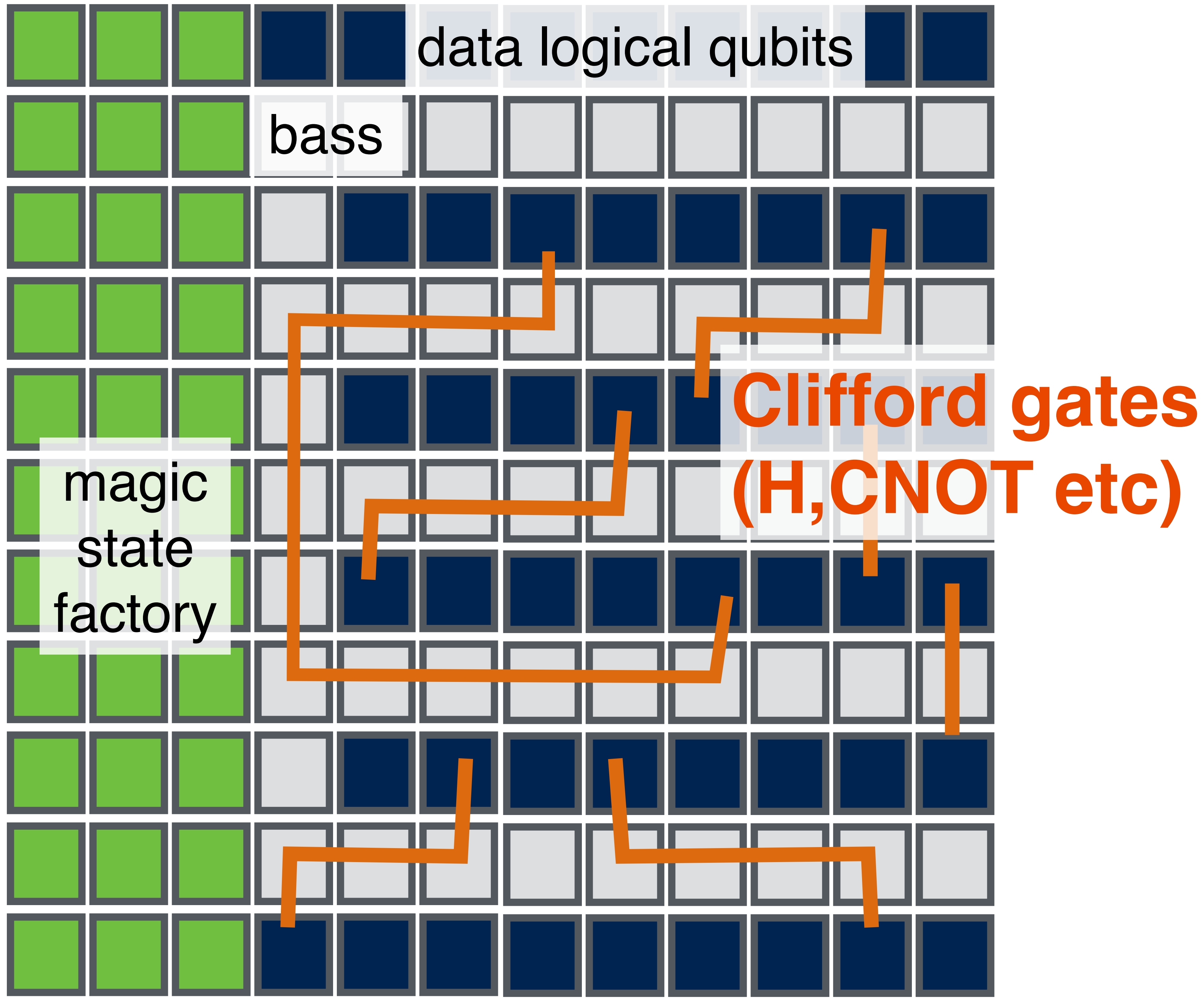

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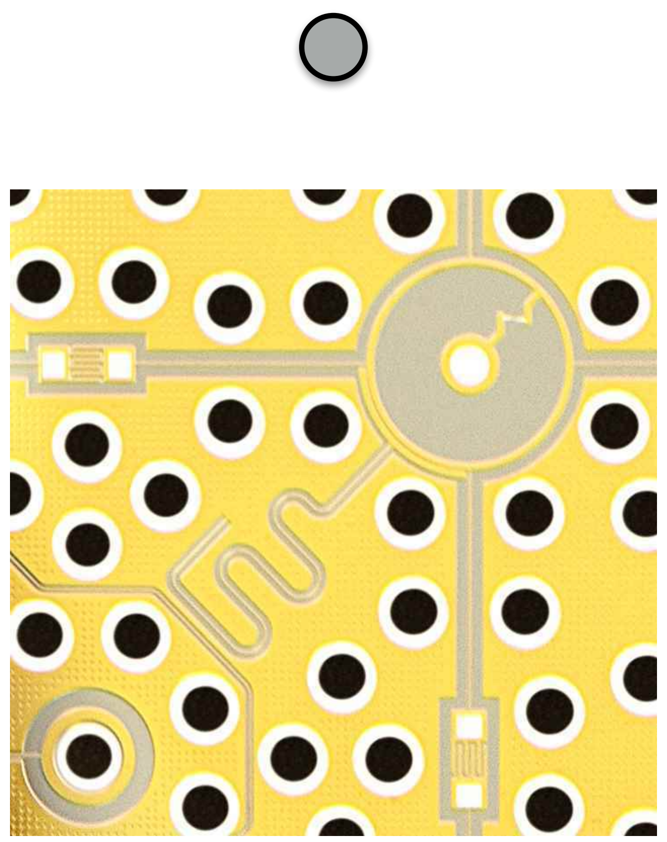

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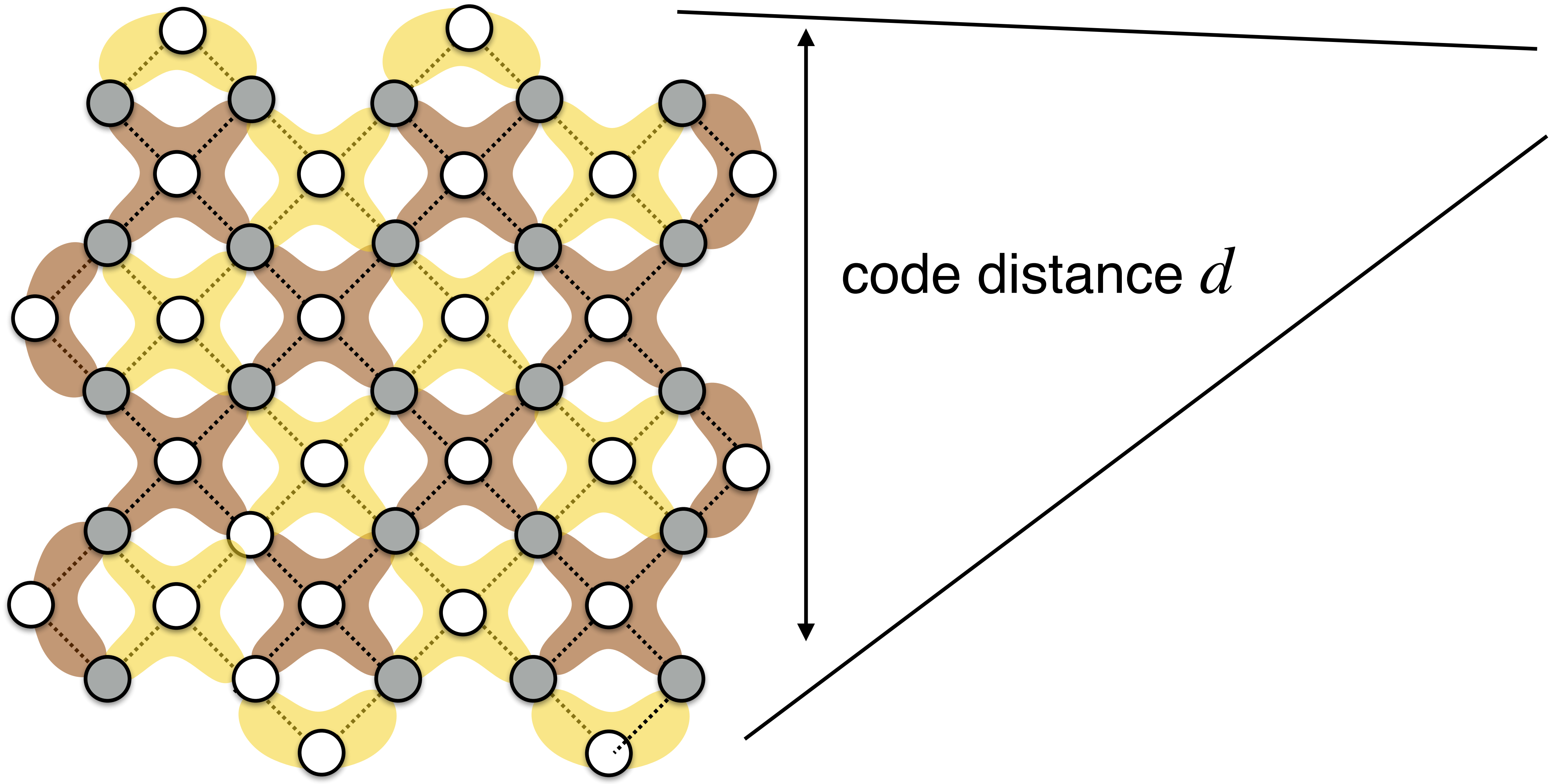

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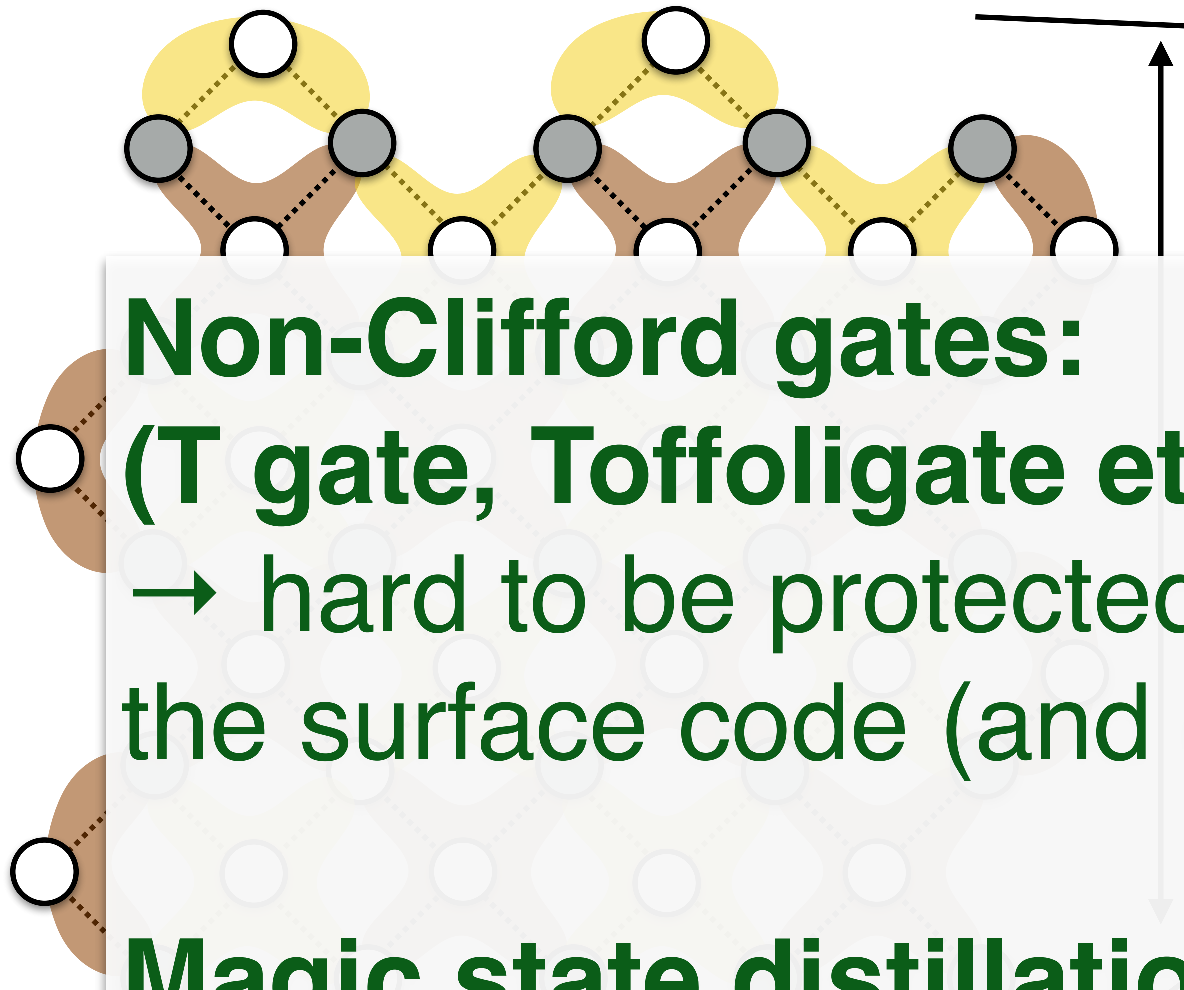

Non-Clifford gates:
(T gate, Toffoligate etc)
→ hard to be protected on the surface code (and other codes).

Magic state distillation:
→ clean magic states should be supplied from factory to do them.

logical qubit: $2d^2-1$ physical qubits

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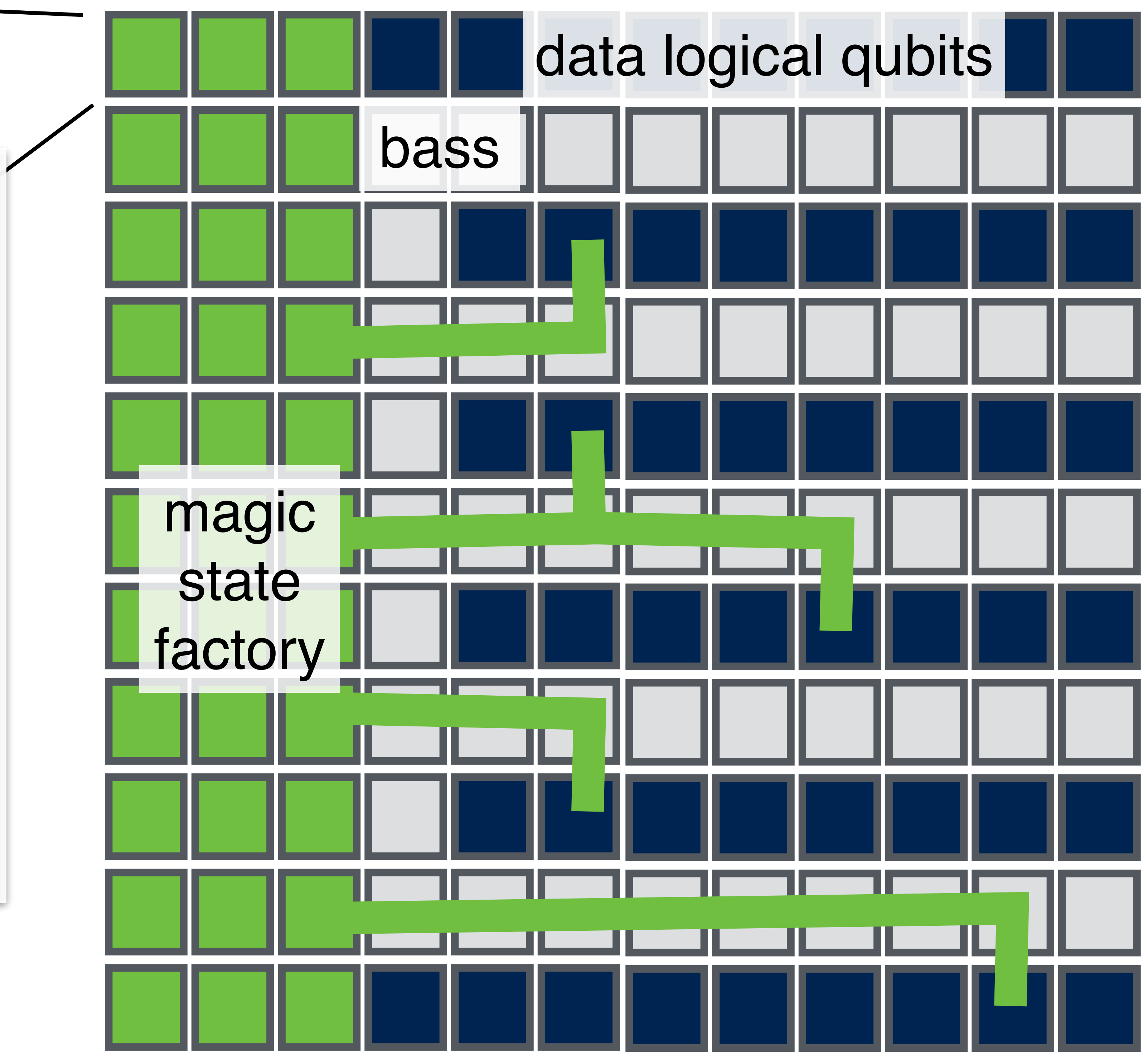

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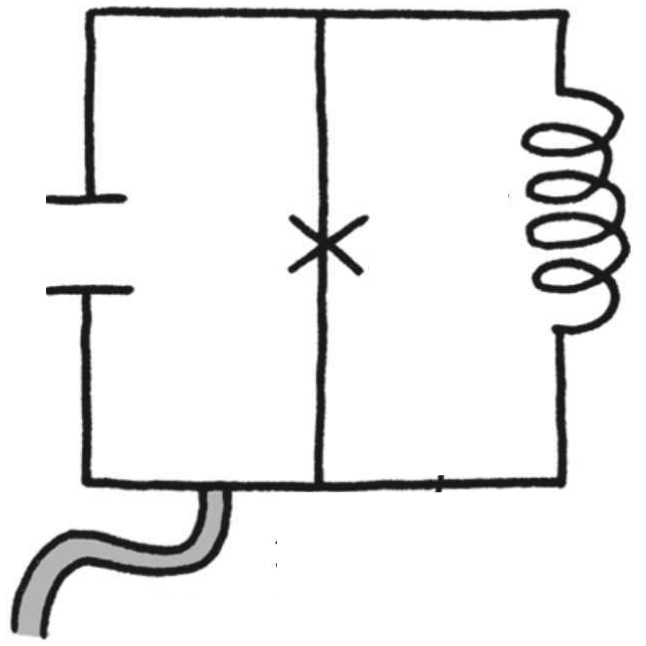
Quantum algorithm using 100-1000 logical qubits

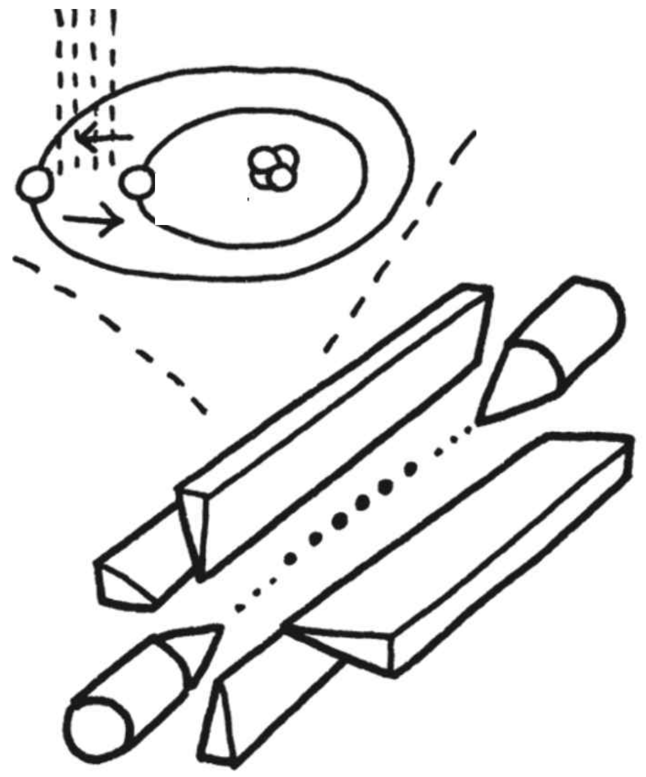
Total amount of physical qubits → 10^6

Layered Structure of Fault-Tolerant Quantum Computing

Superconducting qubit

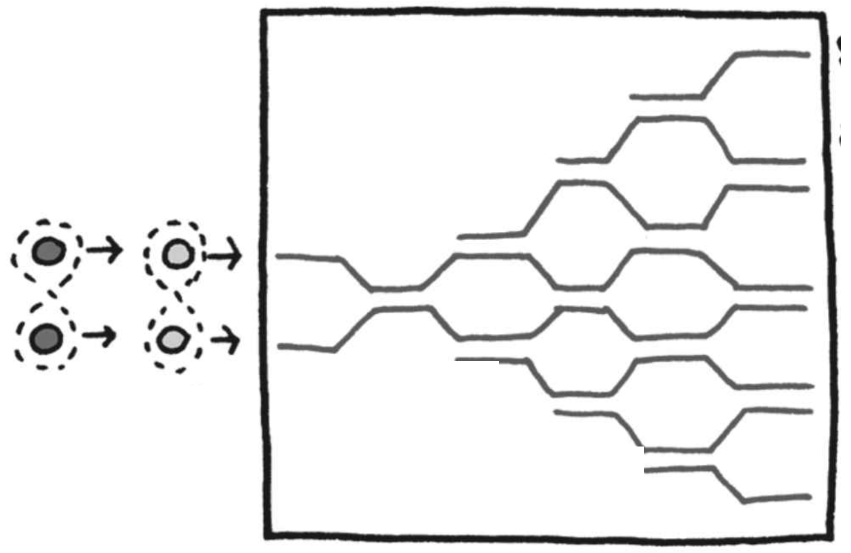


Trapped Ion qubit

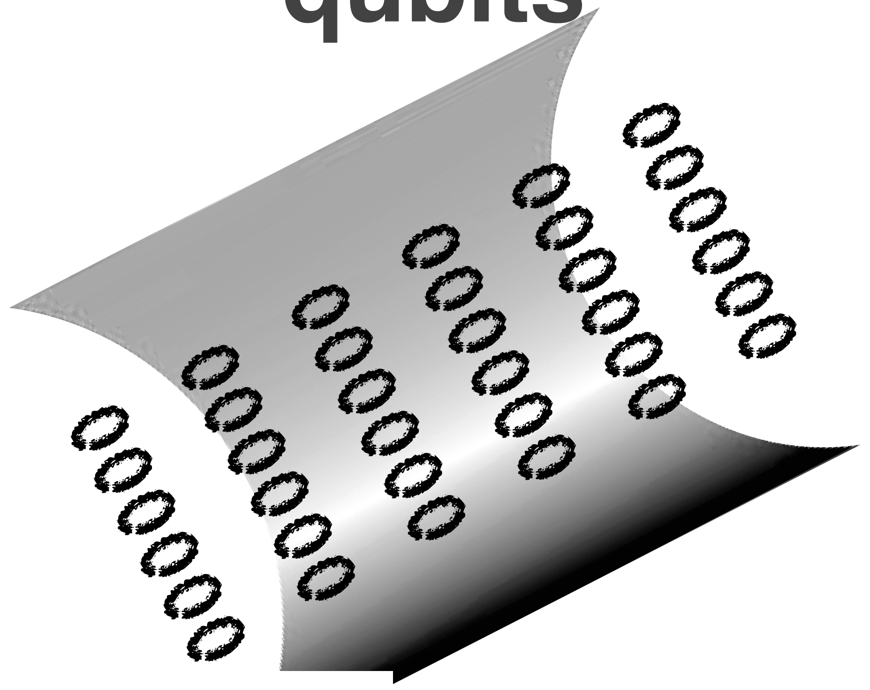


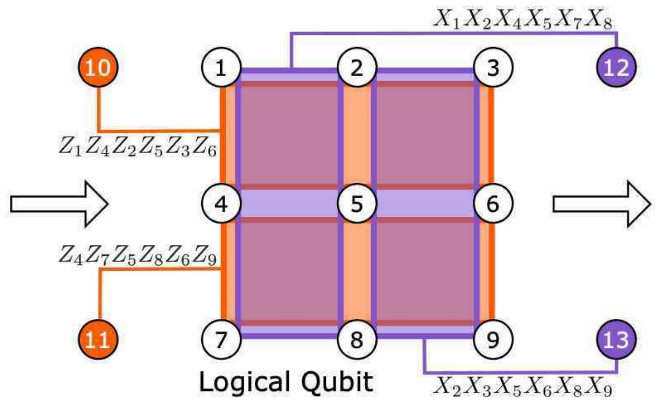
Semiconductor qubit

Photonic qubits

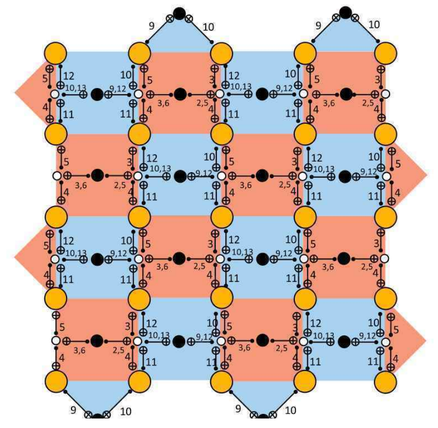


Neutral atom qubits





arXiv:2009.11482



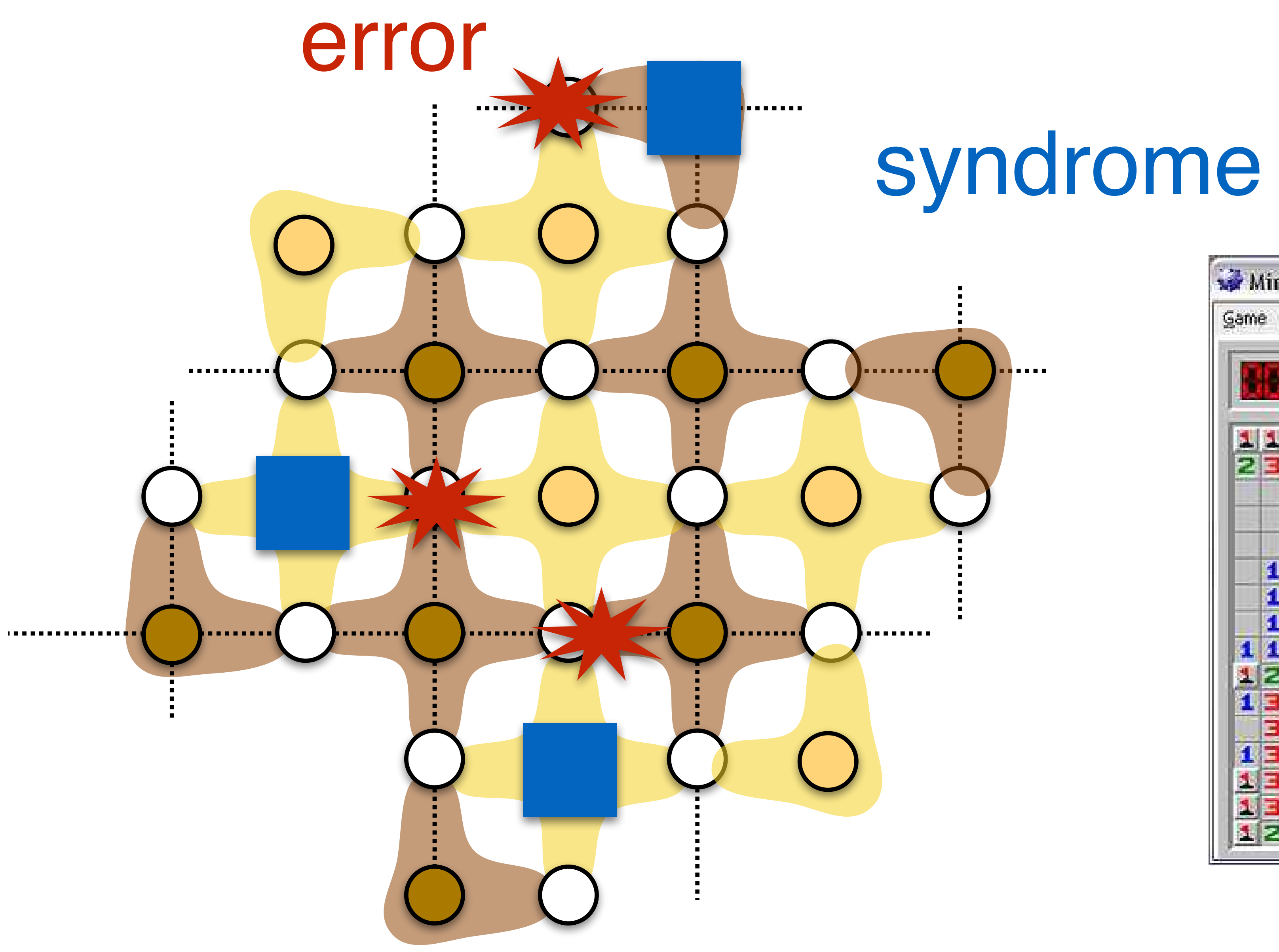
PhysRevX.10.011022

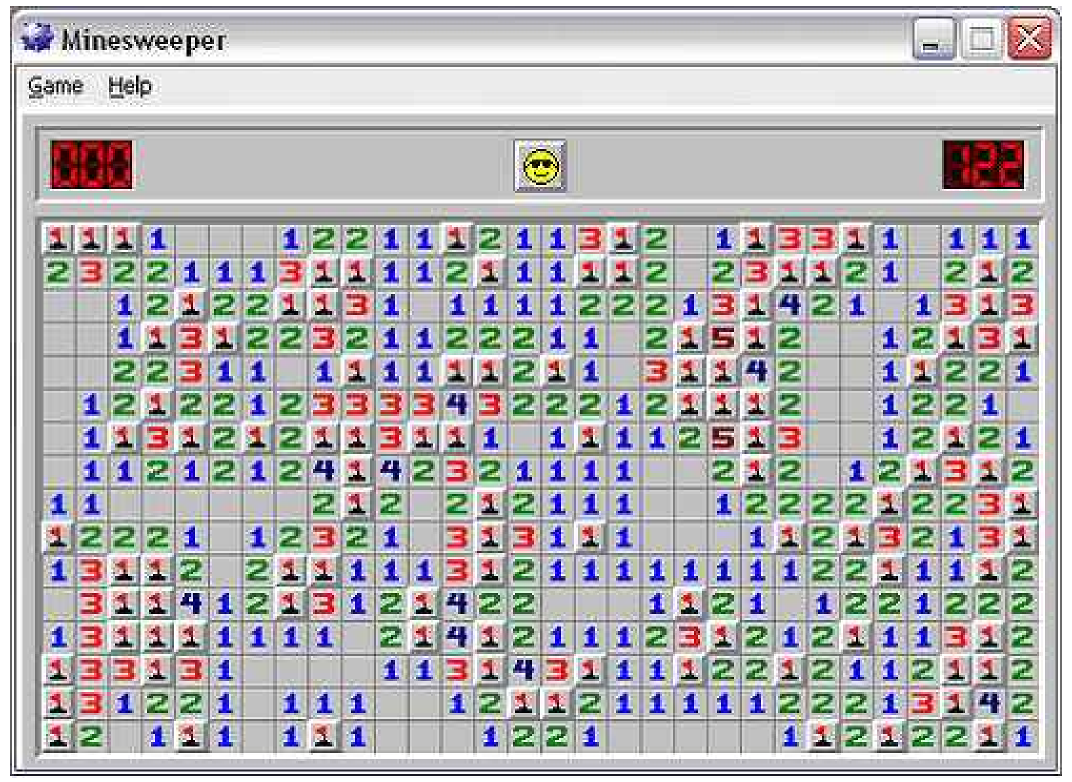
Physical Gate Control

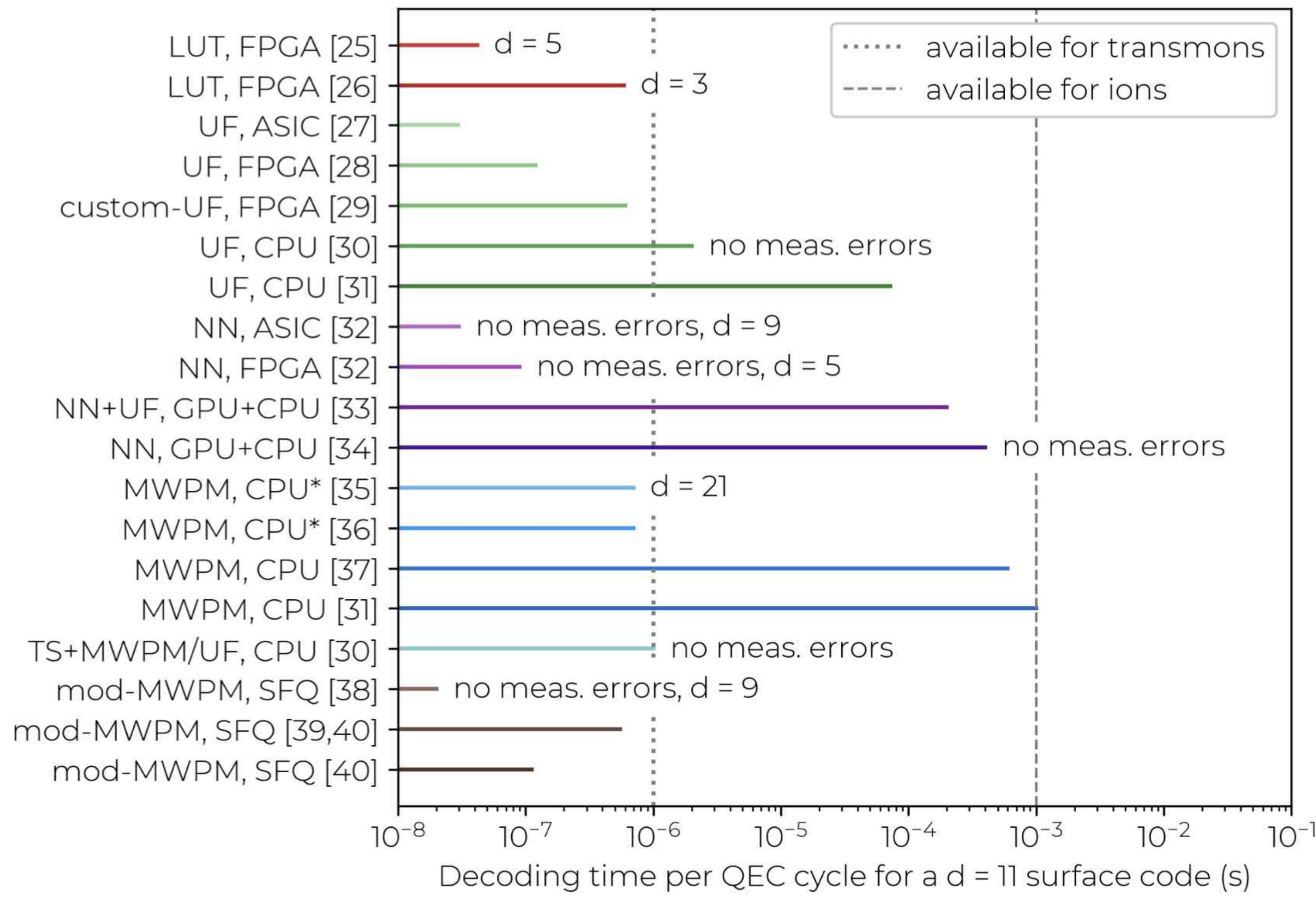
Different physical systems have different constraints on the architecture in terms of how they are controlled.

Quantum Hardware

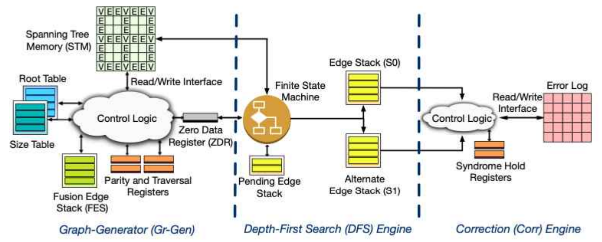
Layered Structure of Fault-Tolerant Quantum Computing







F. Battistel, et al. "Real-Time Decoding for Fault-Tolerant Quantum Computing: Progress, Challenges and Outlook." arXiv (2023)



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Error detection and Correction — Need a real-time decoding algorithm that decode without delay.

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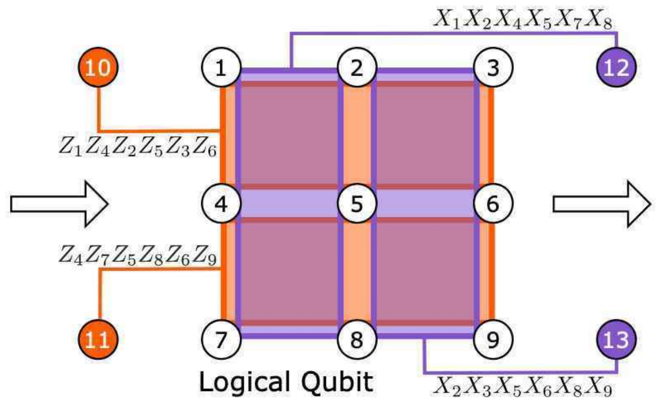

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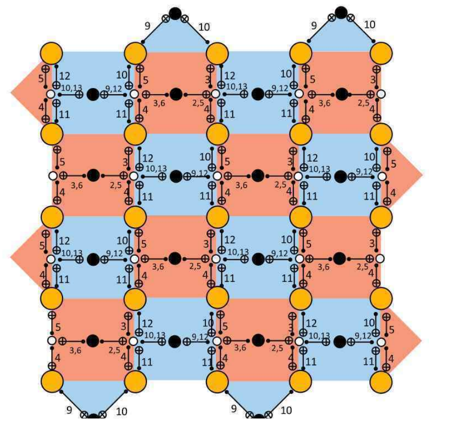

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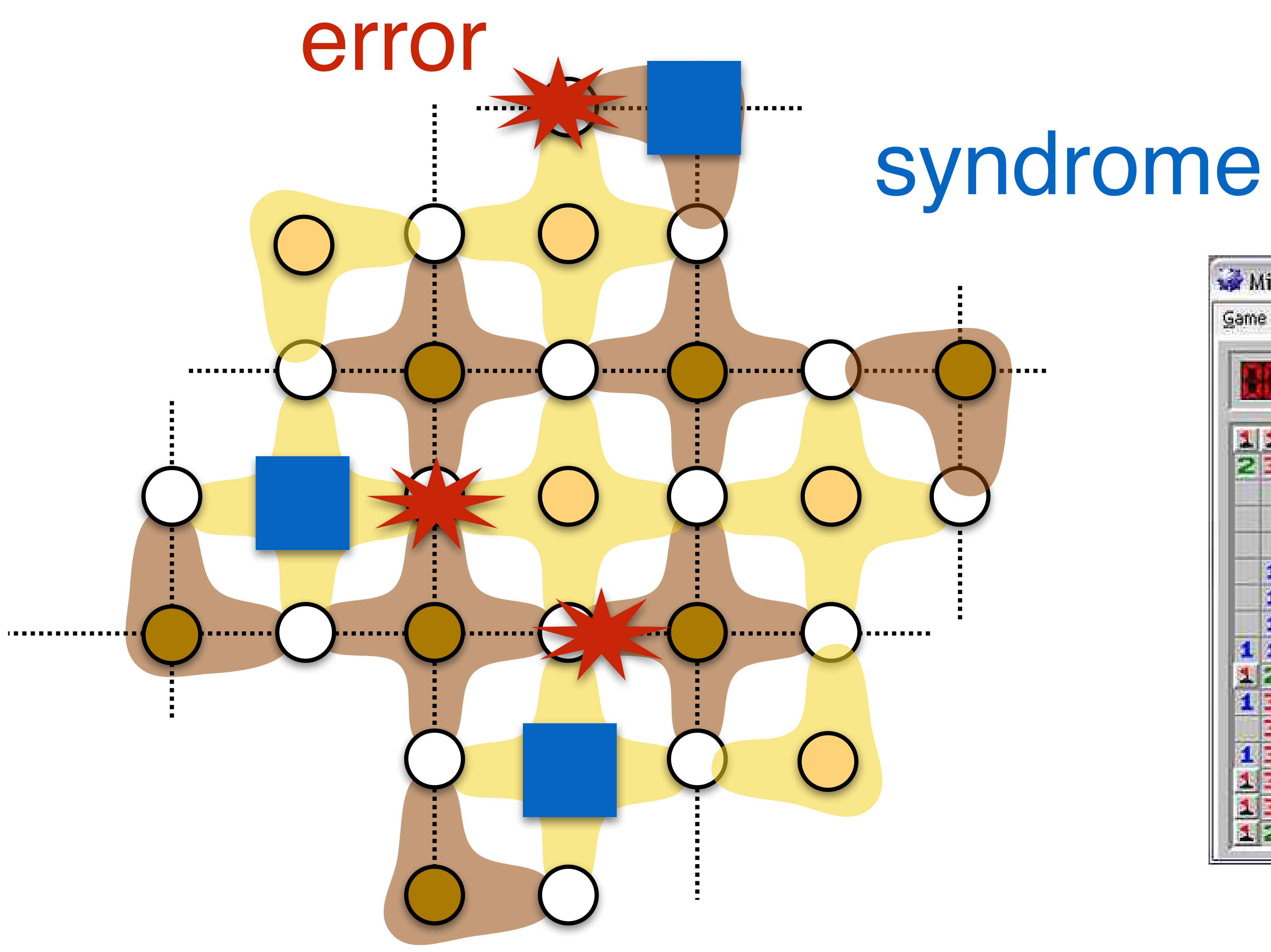
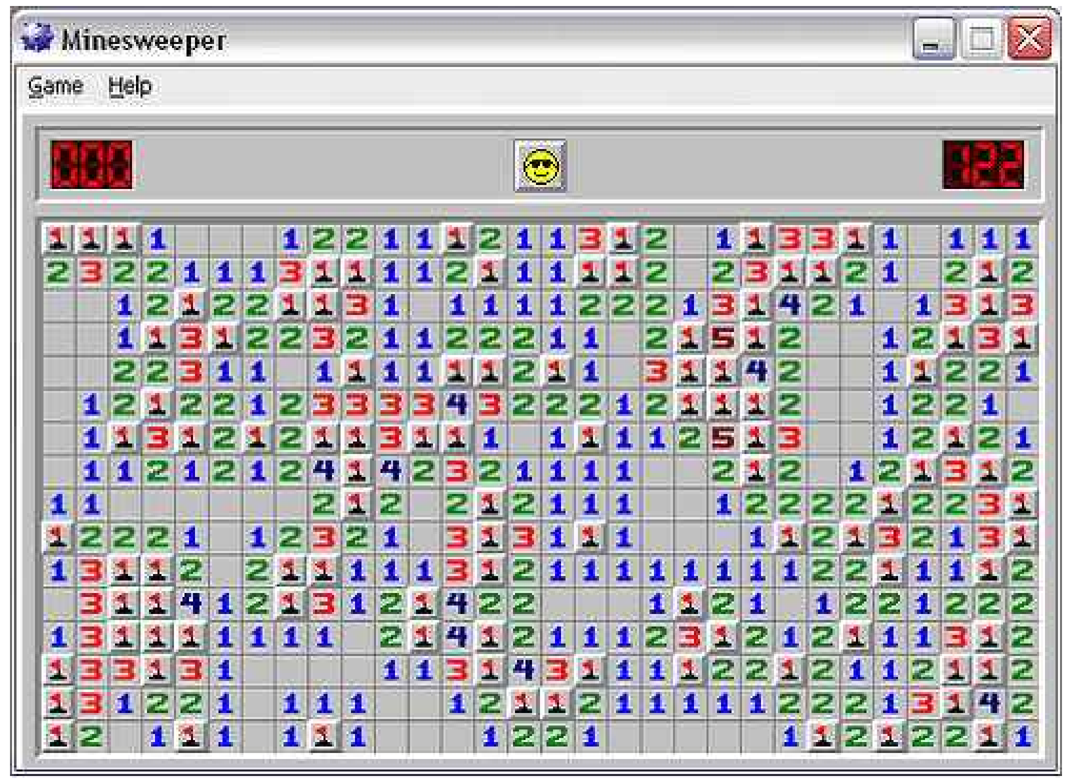
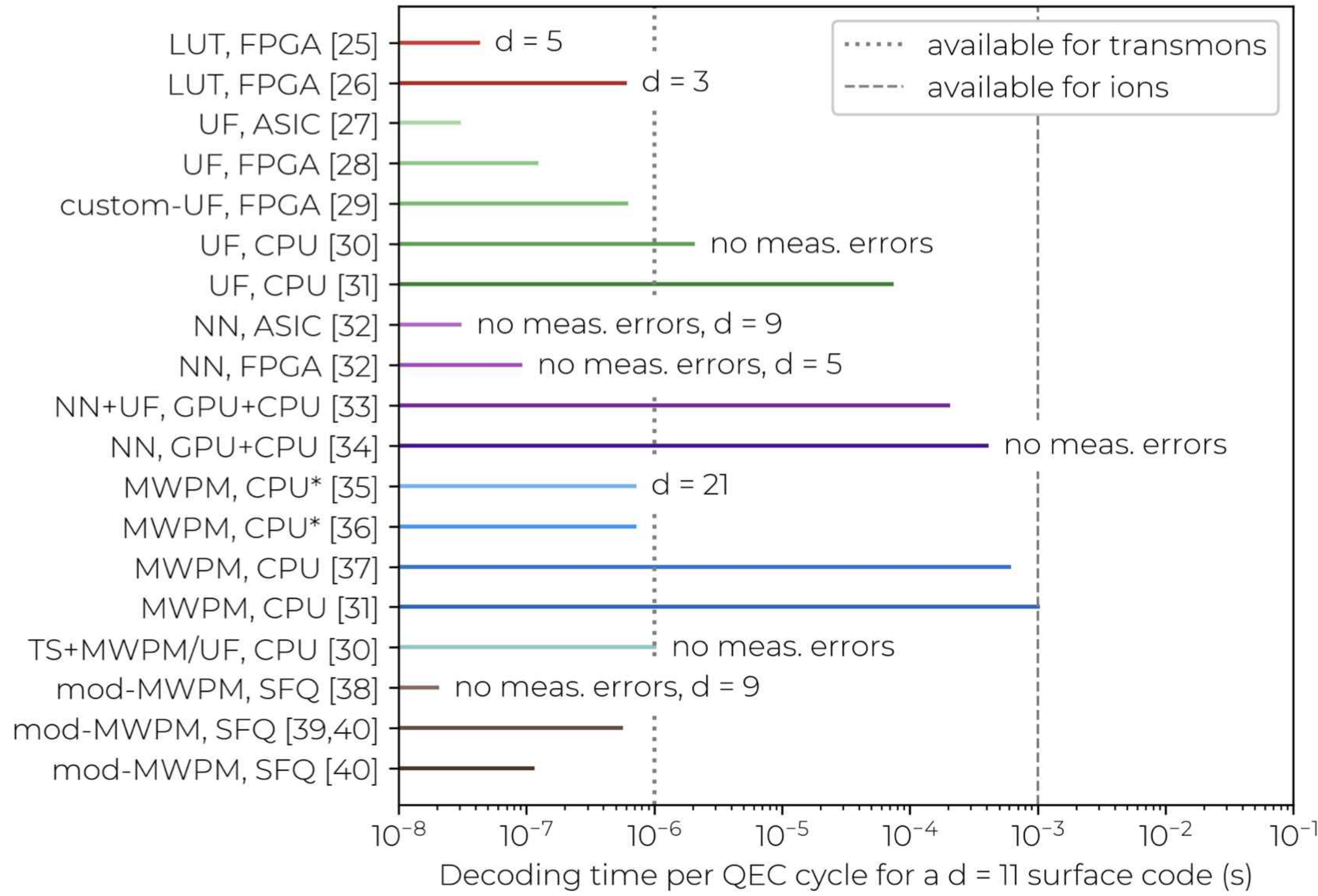
Quantum Hardware

Layered Structure of Fault-Tolerant Quantum Computing







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Quantum Computer Architecture

Fault-Tolerant Logical gates	Lattice surgery, defect braiding…
Error detection and Correction	Need a real-time decoding algorithm that decode without delay.
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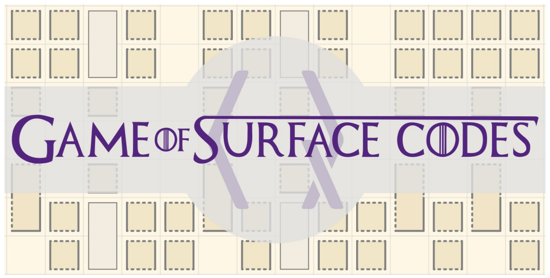
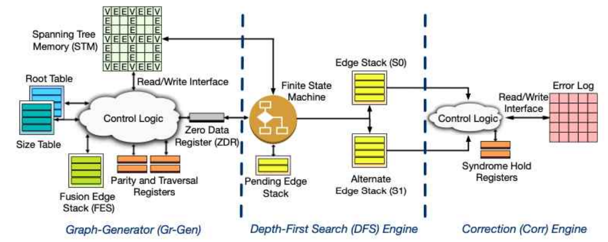
Quantum Hardware



Quantum 3, 128 (2019).



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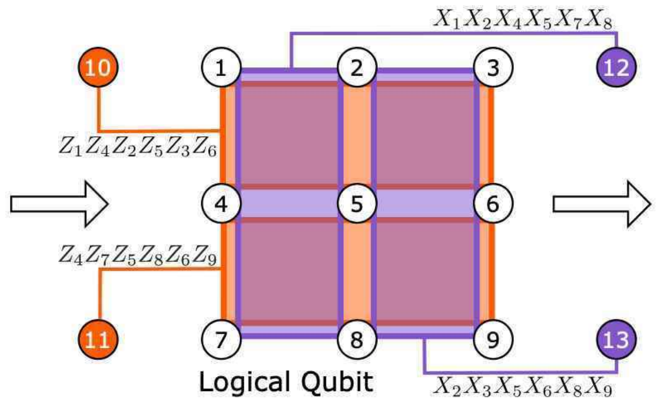

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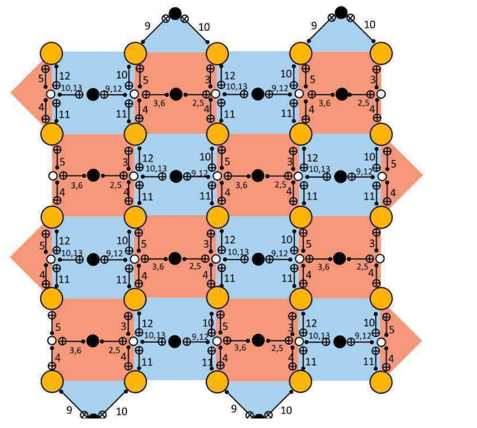

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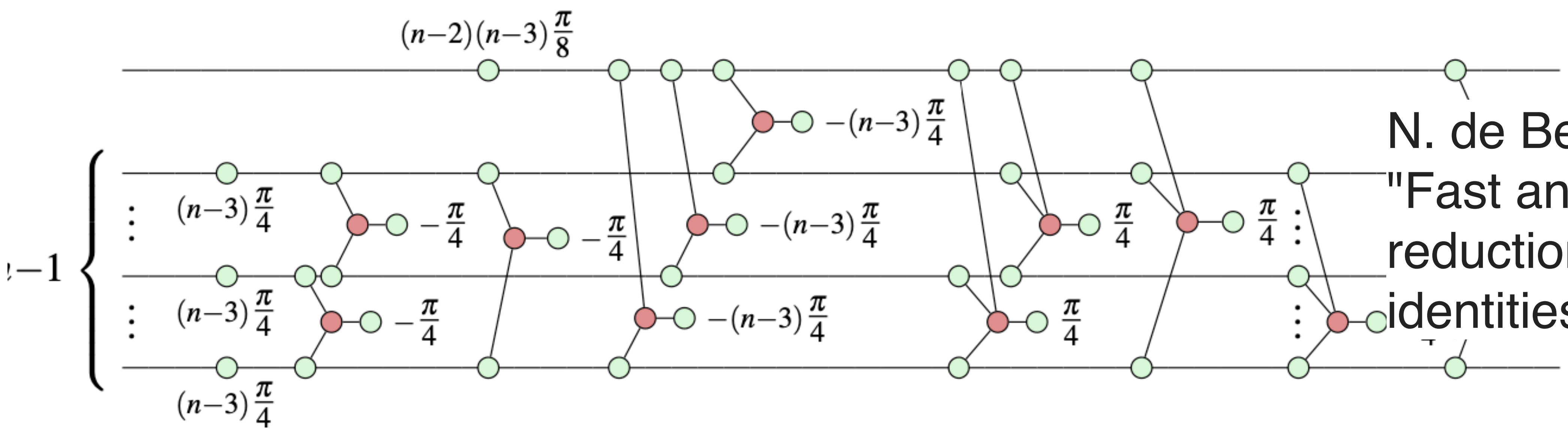


N. de Beaudrap, X. Bian, and Q. Wang. "Fast and effective techniques for T-count reduction via spider nest identities." arXiv (2020).

Compilation and Optimization

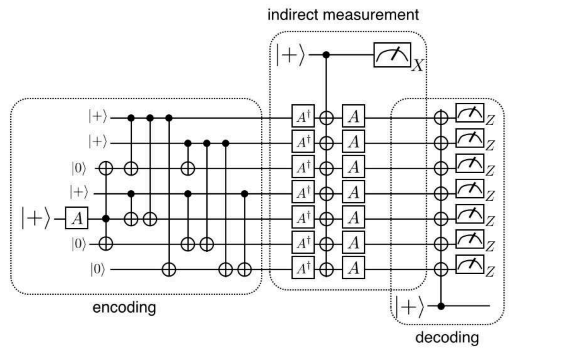
Logical Quantum Circuits

Decomposing and optimizing Clifford+T circuits.

Digital Quantum Computer

Magic state distillation

Special magic state preparation to protect T-gates from errors, which is essential for universal computation.



Quantum Computer Architecture

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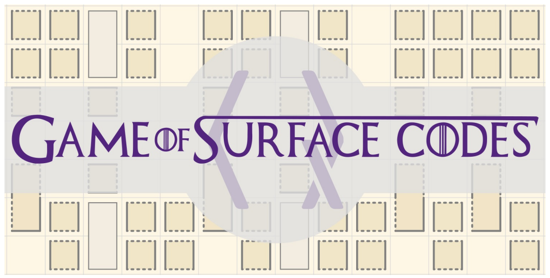

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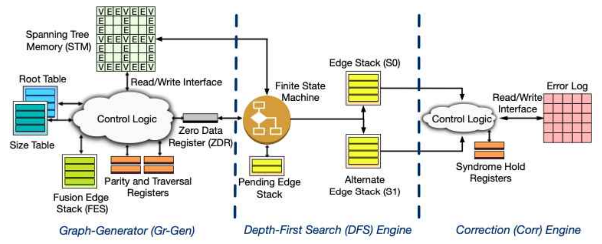

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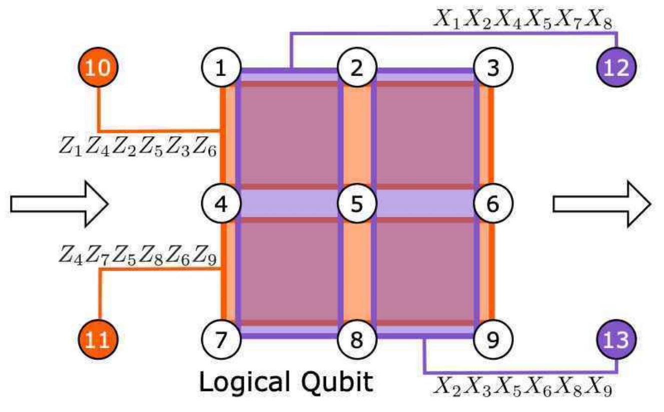

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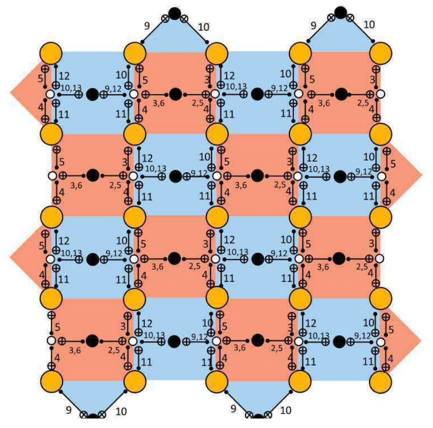

PhysRevX.10.011022

Quantum Hardware

Layered Structure of Fault-Tolerant Quantum Computing

Application

Quantum algorithm

Factoring, Machine learning, Quantum Chemistry, Condensed Matter Physics

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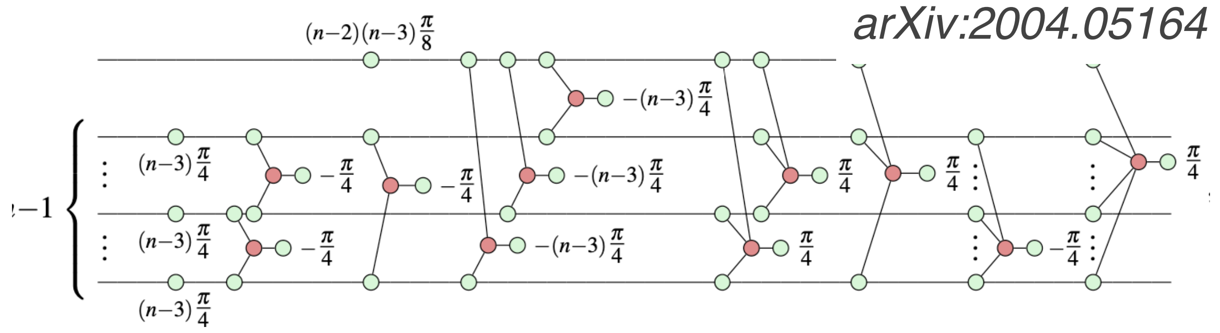

arXiv:2004.05164

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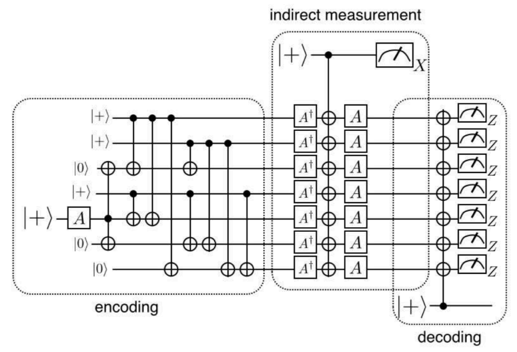

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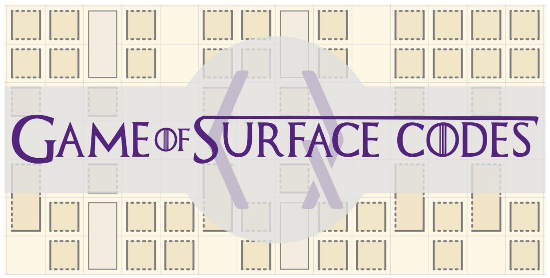

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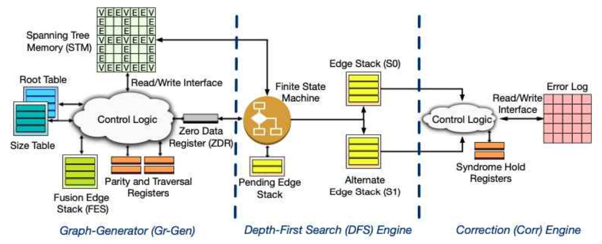

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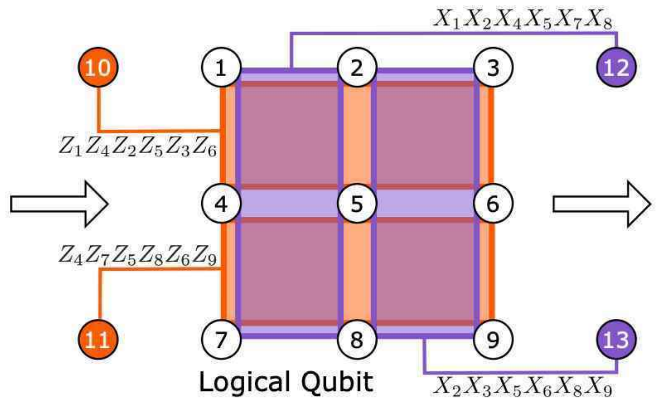

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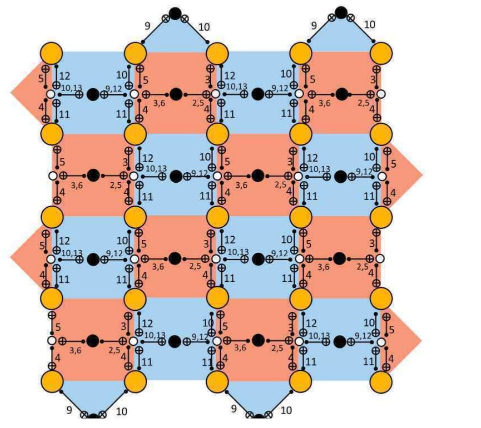

PhysRevX.10.011022

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Quantum Hardware

Resource estimate for solving classically intractable problems

Prime Factorization (Security Impact)

Prime factorization of N bits ⟶ O(N^3) Toffoli gate

2048bit ⟶ ~10^{10} gates

Error rate 0.1%, 20M qubits (~10^4/logic qubit), 8 hours (1 µsec code cycle)

"How to factor 2048 bit RSA integers in 8 hours using 20 million noisy qubits" Gidney-Ekera arXiv:1905.09749

(classical：RSA-250 829bit, 2700 cores year Boudot et al 2020.2.28)

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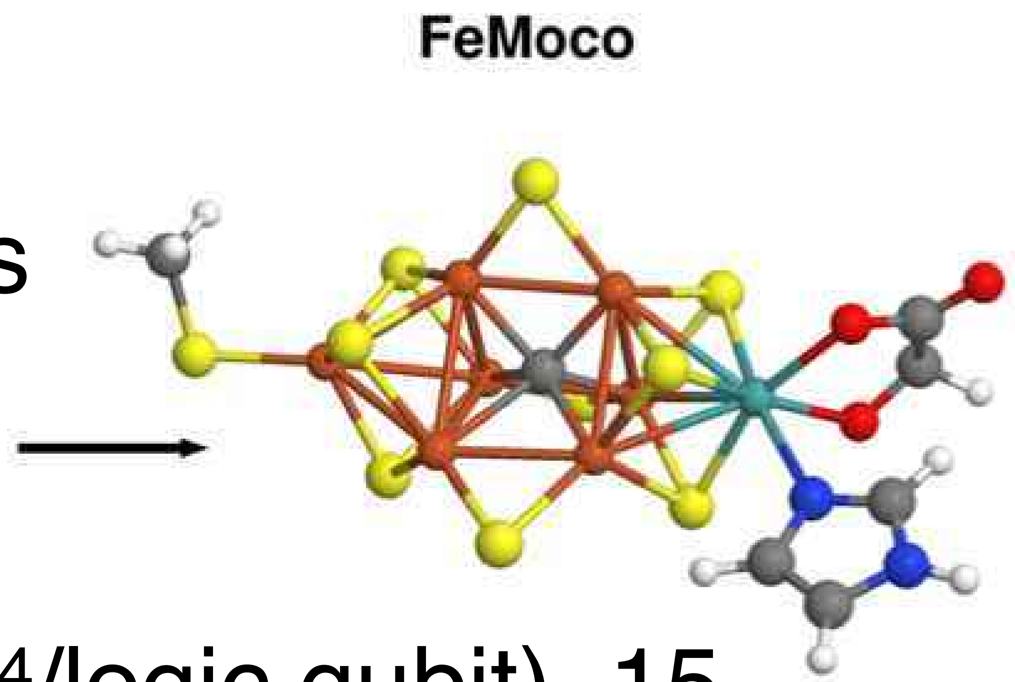
Quantum Chemistry

FeMoco, 54 electrons, 54 orbitals
0.1-1mHa,
⟶ 10^{15} Tgate



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“Elucidating reaction mechanisms on quantum computers”Reiher et al, PNAS 2017

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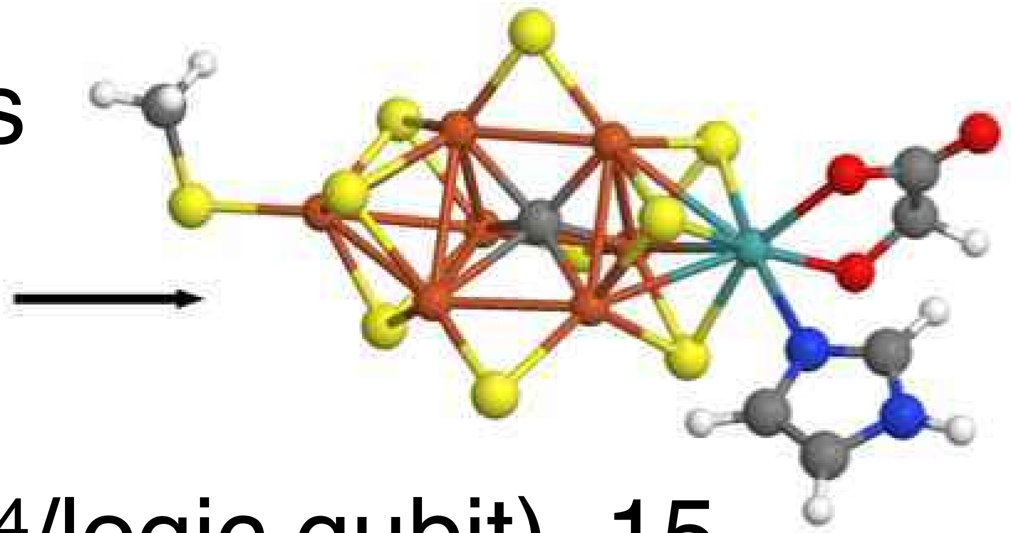

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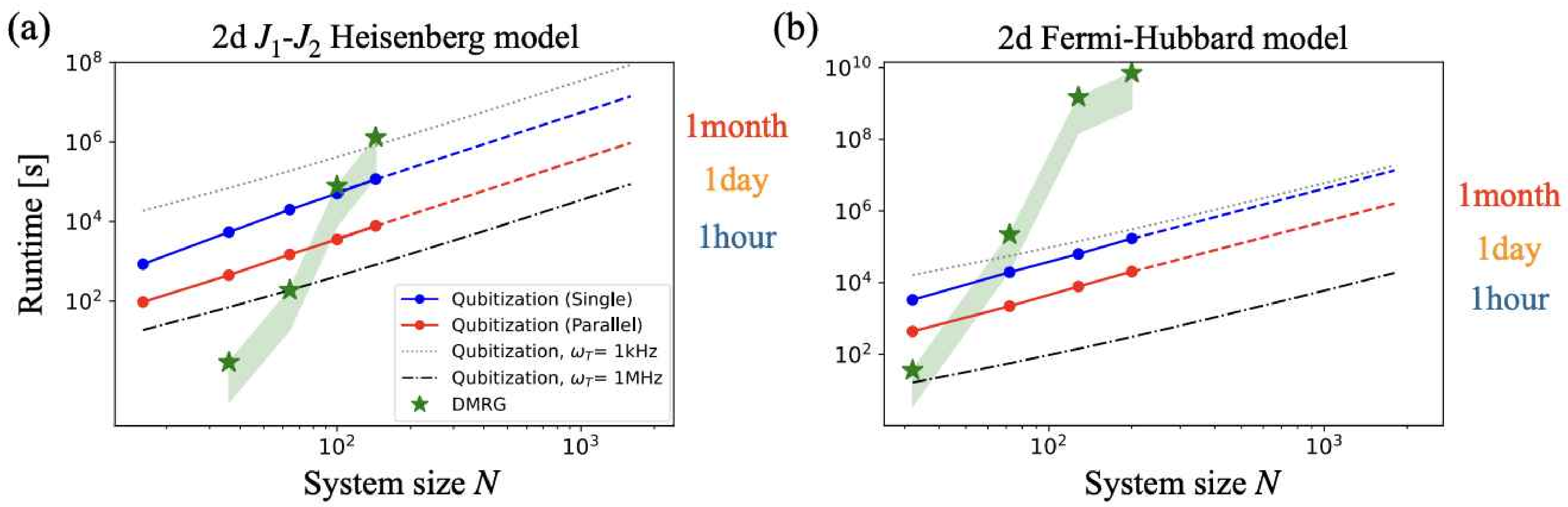
Condensed Matter Physics

Heisenberg model and Fermi-Hubbard model



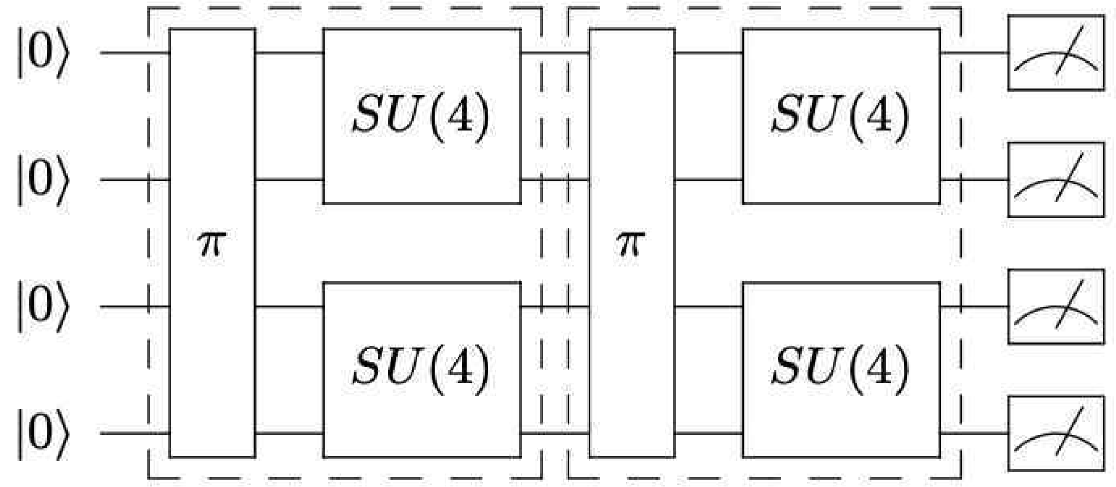
Error rate 0.1%, ~ 0.5M qubits (~10^3/logic qubit), hours (1µsec code cycle)

early FTQC era?

Quantum Chasm: A large gap between NISQ and FTQC

- State-of-the-art quantum computer can perform tasks that are intractable for classical computer (*random quantum circuit sampling*).
- Still quantum advantage in a meaningful and useful tasks is missing. *(Can we obtain a quantum volume advantage?)*
- Current state-of-the-art NISQ devices are reaching over 100 qubits. Yet, Quantum Volume is very limited.
- On the other side, fully fledged fault-tolerant quantum computer needs 1 million physical qubits…

Quantum Volume
[A. W. Cross et al. PRA, 2019]:



$$m^2(1.29\sqrt{m} - 0.78)p < 1,$$

m: linear length of square grid
p: error rate

p=10^{-4} → $\log_2$(Quantum Volume) = 37

Still, fully simulatable on HPC!

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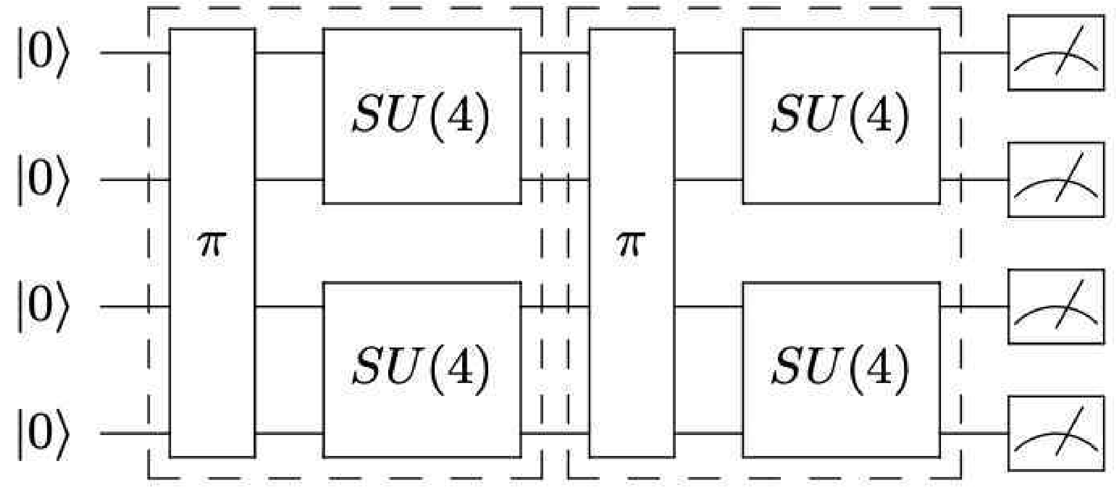

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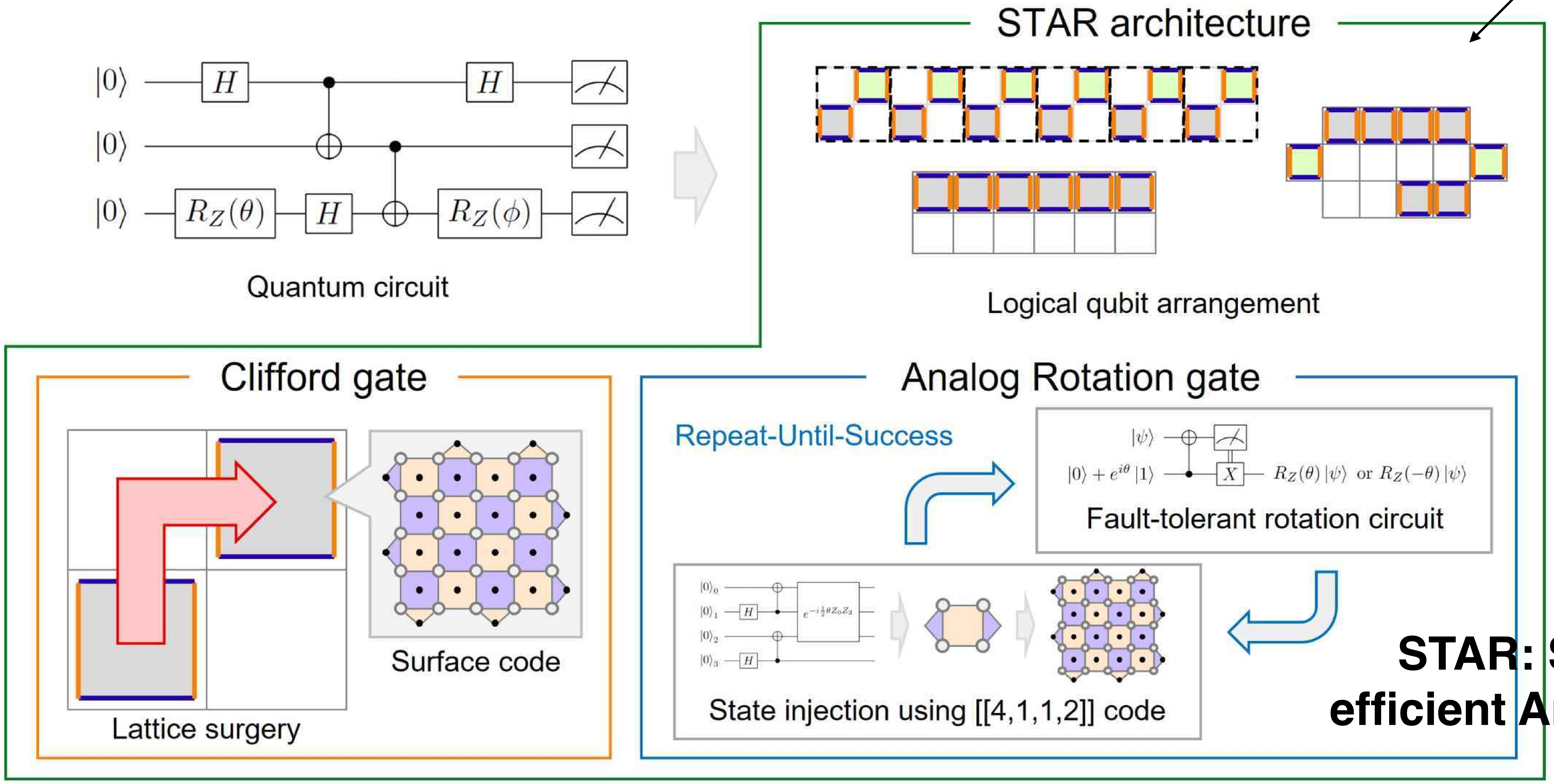
There is a large gap between NISQ (meaningfully exploited # of qubits) and FTQC (required # of qubits).

Partially Fault-tolerant Quantum Computing Architecture

Y Akahoshi, K Maruyama, H Oshima, S Sato, K Fujii arXiv preprint arXiv:2303.13181

Planar surface code

Assumption: 10,000 qubits with 10^{-4} error rate.



STAR: Space-Time efficient Analog Rotation

Clifford gates are fully protected.

1Q rotations are not fully protected, but fairly high fidelity.

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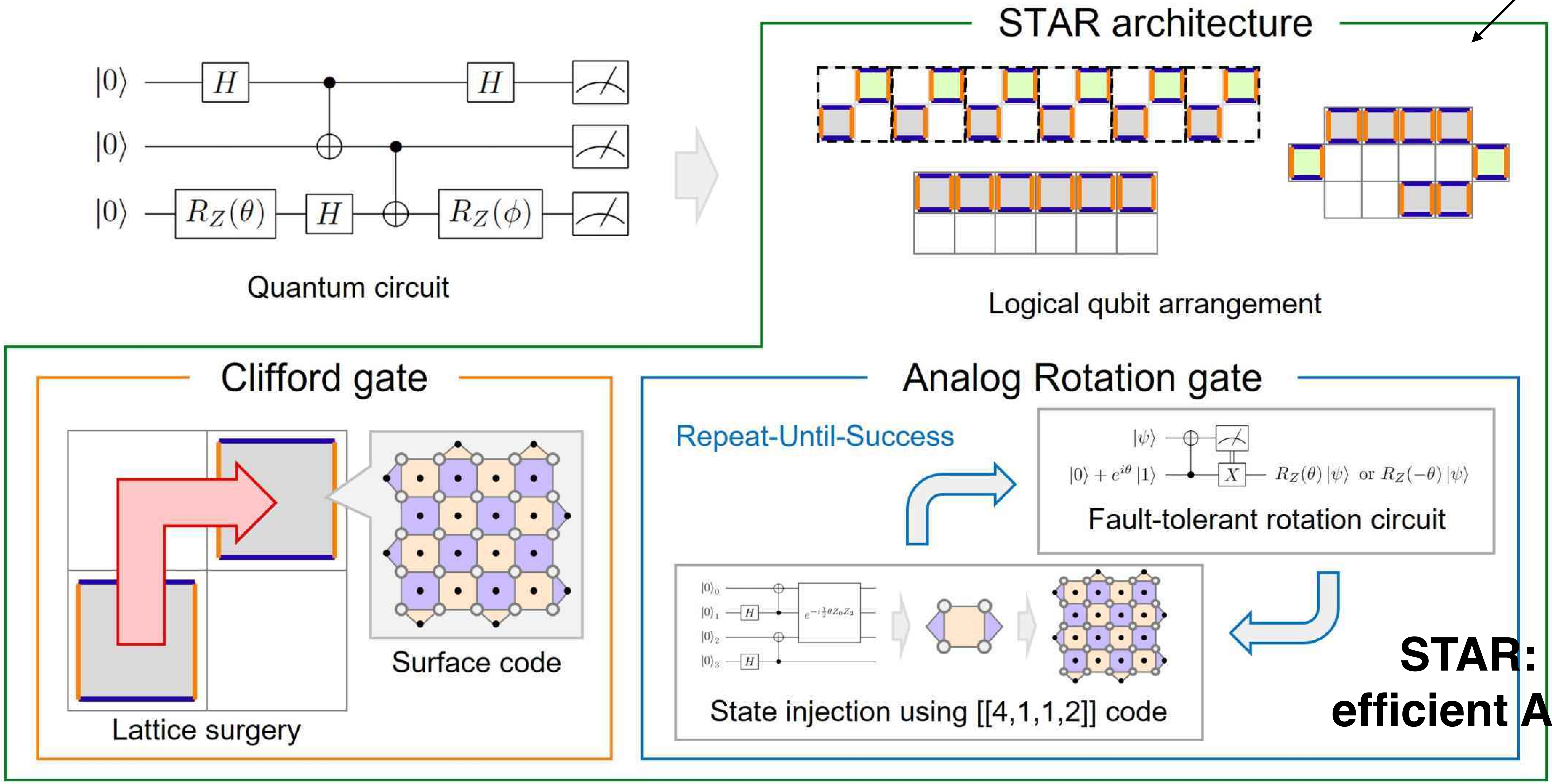

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Quantum Volume of 2^64 can be achievable with 10,000 qubits with 10^{-4} error rate.