Limit theorem for QSDEs; cavity QED relay model

In this set of notes we take a very brief look at the use of quantum stochastic differential equations (QSDEs) to model open quantum systems, and also at a very useful theorem for QSDEs that allows us to obtain highly simplified component models for certain cavity QED systems in the ***small volume limit***. We illustrate the use of this theorem in the derivation of a simple scattering matrix model for a cavity QED relay, which we will use later in our discussion of continuous quantum error correction.

General references for today's material include, on QSDE modeling of quantum optical systems:

- Supplementary Material of J. Kerckhoff *et al.*, "Designing Quantum Memories with Embedded Control: Photonic Circuits for Autonomous Quantum Error Correction," Phys. Rev. Lett. **105**, 040502 (2010),

on limit theorems for QSDE models:

- L. Bouten, R. van Handel and A. Silberfarb, "Approximation and limit theorems for quantum stochastic models with unbounded coefficients," Journal of Functional Analysis **254**, 3123 (2008),

and on the cavity QED relay model (a slightly more complex version than the one we consider here):

- H. Mabuchi, "Cavity-QED models of switches for attojoule-scale nanophotonic logic," Phys. Rev. A **80**, 045802 (2009).

Preliminaries: Schrödinger evolution operator, Heisenberg picture, SDEs

The Schrödinger evolution operator $U(t_f, t_i)$ is defined by

$$|\psi(t_f)\rangle = U(t_f, t_i)|\psi(t_i)\rangle,$$

where the state must satisfy the Schrödinger Equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

Hence we find the evolution equation

$$i\hbar\frac{d}{dt}(U(t,t_i)|\psi(t_i)\rangle) = HU(t,t_i)|\psi(t_i)\rangle,$$

$$i\hbar\left(\frac{d}{dt}U(t,t_i)\right)|\psi(t_i)\rangle = HU(t,t_i)|\psi(t_i)\rangle,$$

$$\frac{d}{dt}U(t,t_i) = \frac{1}{i\hbar}HU(t,t_i), \quad U(t_i,t_i) = 1.$$

If the Hamiltonian is constant this can be solved straightforwardly via the operator exponential,

$$U(t,t_i) = \exp(-iH(t-t_i)/\hbar).$$

We note that any time-dependent operator moment can thus be written

$$\langle O\rangle_t = \langle\psi(t)|O|\psi(t)\rangle = \langle\psi(t_i)|U^\dagger(t,t_i)OU(t,t_i)|\psi(t_i)\rangle,$$

and we note that we can reproduce all such observable quantities in the Heisenberg picture where the state vector $|\psi(t_i)\rangle$ is considered to be constant (held to its initial condition) while the operators evolve according to

$$O(t) = U^\dagger(t,t_i)OU(t,t_i).$$

In quantum field theories it is generally more convenient to work in the Heisenberg picture than the Schrödinger picture.

Below we will have a quick look at some quantum input-output models based on quantum stochastic differential equations (QSDEs), which can be thought of as non-commutative generalizations of classical stochastic differential equations (SDEs) which are commonly used in engineering, physics, and mathematical finance. In both the quantum and classical cases, stochastic differential equations come in two different flavors: Stratonovich and Itô. When using Itô SDE's one must be careful to observe the Itô Rule, which says (in the classical case) that if x_t obeys the Itô SDE

$$dx_t = A(x_t)dt + B(x_t)dW_t,$$

then a variable y_t related to x_t via

$$y_t = f(x_t)$$

evolves according to

$$dy_t = \left[A(x_t)\frac{\partial f}{\partial x} + \frac{1}{2}B^2(x_t)\frac{\partial^2 f}{\partial x^2} \right]dt + B(x_t)\frac{\partial f}{\partial x}dW_t,$$

where the second-derivative term in the square brackets is known as the Itô correction. We can understand this as corresponding to a type of Taylor expansion in which we keep terms to *second* order and then apply the rules $dW_t^2 = dt,\ dW_t dt = dt^2 = 0.$ Note that if f is a linear function the Itô correction vanishes and we recover the prediction of normal calculus. An important advantage of working with Itô SDE's is that if x_t obeys the Itô SDE

$$dx_t = A(x_t)dt + B(x_t)dW_t,$$

then x_t is uncorrelated with $dW_t.$ This considerably simplifies the computation of statistical moments. For example consider the linear SDE model

$$dx_t = Ax_t dt + FdV_t,$$

with x_t a scalar and $A < 0$ (the Ornstein-Uhlenbeck model). We then have

$$\begin{aligned} d\langle x_t\rangle &= A\langle x_t\rangle dt + F\langle dV_t\rangle \\ &= A\langle x_t\rangle dt, \\ \langle x_t\rangle &= \langle x_0\rangle \exp(At), \end{aligned}$$

and if $y_t = x_t^2$, so that $\langle y_t\rangle$ is the variance of x_t,

$$\begin{aligned} dy_t &= [2Ax_t^2 + F^2]dt + 2Fx_t dV_t, \\ d\langle y_t\rangle &= [2A\langle y_t\rangle + F^2]dt + 2F\langle x_t dV_t\rangle \\ &= [2A\langle y_t\rangle + F^2]dt + 2F\langle x_t\rangle\langle dV_t\rangle \\ &= [2A\langle y_t\rangle + F^2]dt, \\ \langle y_t\rangle &= \exp(2At)\left\{\langle y_0\rangle + \int_0^t ds \exp(-2As)F^2\right\} \\ &= \exp(2At)\left\{\langle y_0\rangle + F^2\int_0^t ds \exp(-2As)\right\}. \end{aligned}$$

If we assume that x_t evolves from a known value x_0 at $t = 0$, then $\langle x_0\rangle = x_0$ and $\langle y_0\rangle = x_0^2$, and the mean-square uncertainty in x_t is

$$\begin{aligned} \langle x_t^2\rangle - \langle x_t\rangle^2 &= \langle y_t\rangle - \langle x_t\rangle^2 \\ &= \exp(2At)F^2\int_0^t ds \exp(-2As) \\ &= \exp(2At)F^2\left(-\frac{1}{2A}\right)(\exp(-2At) - 1) \\ &= -\frac{F^2}{2A}(1 - \exp(2At)). \end{aligned}$$

The mean-square uncertainty thus has a steady-state value as $t \to \infty$ (assuming $A < 0$)

$$\langle x_t^2\rangle - \langle x_t\rangle^2 \to \frac{F^2}{2|A|}.$$

There are quantum generalizations of the above 'stochastic calculus' methods, which underly the derivations presented below, but we will not have time to go into these. A key point of similarity is that we will generally need to keep second-order terms in differential expressions and apply quantum generalizations of the rule $dW_t^2 = dt.$

QSDE models of open quantum systems

In the Heisenberg picture we can consider a unitary evolution operator U_t such that observables evolve according to

$$a_t = U_t a_0 U_t^*,$$

which has the differential form

$$da_t = U_t a_0 dU_t^* + dU_t a_0 U_t^* + dU_t a_0 dU_t^*.$$

This evolution operator (which we note is actually the hermitian conjugate of the usual Schrödinger evolution operator) is assumed to obey a 'left' QSDE of the Hudson-Parthasarathy form,

$$dU_t = U_t\left\{\sum_{i,j=1}^{n}(N_{ij}-\delta_{ij})d\Lambda_t^{ij} + \sum_{i=1}^{n} M_i(dA_t^i)^* + \sum_{i=1}^{n} L_i\,dA_t^i + K\,dt\right\},$$

where it is required that

$$K+K^* = -\sum_{i=1}^{n} L_i L_i^*, \quad M_i = -\sum_{j=1}^{n} N_{ij}L_j^*, \quad \sum_{j=1}^{n} N_{mj}N_{lj}^* = \sum_{j=1}^{n} N_{jm}^* N_{jl} = \delta_{ml}.$$

We see from these conditions that the M_i are actually determined by N_{ij} and the L_j, so such a QSDE is actually fully specified by $\{K, L_i, N_{ij}\}$.

With the abstract notation

$$dU_t = U_t\{F\}, \quad dU_t^* = \{F^*\}U_t^*,$$

we thus see that

$$da_t = U_t\,a_0\,dU_t^* + dU_t\,a_0\,U_t^* + dU_t\,a_0\,dU_t^* = U_t\,a_0\{F^*\}U_t^* + U_t\{F\}a_0U_t^* + U_t\{F\}a_0\{F^*\}U_t^*.$$

Following Gardiner's method for deriving the master equation, we can write (averaging over the noise terms)

$$d\langle a_t\rangle = \langle da_t\rangle = \langle U_t\,a_0\,dU_t^* + dU_t\,a_0\,U_t^* + dU_t\,a_0\,dU_t^*\rangle,$$

$$\frac{d}{dt}\langle a_t\rangle = \mathrm{Tr}\left[\left\{U_t a_0 K^* U_t^* + U_t K a_0 U_t^* + \sum_{i=1}^{n} U_t L_i a_0 L_i^* U_t^*\right\}\rho_0\right],$$

and combining this with

$$\frac{d}{dt}\langle a\rangle = \mathrm{Tr}\left[a_0\frac{d\rho_t}{dt}\right], \quad \rho_t = U_t^*\rho_0 U_t,$$

we obtain

$$\frac{d\rho_t}{dt} = K^*\rho_t + \rho_t K + \sum_{i=1}^{n} L_i^*\rho_t L_i.$$

We should be careful to note that we are here working with 'left' QSDEs and that the L_i operators that have been written above correspond to the hermitian conjugates of the components of the coupling vector that we defined last time in our discussion of (S,L,H) models. The difference arises because the Gough and James network calculus is conventionally derived from 'right' QSDEs for the Schrödinger propagator $dV_t = dU_t^*$, whereas the QSDE limit theorem we want to discuss today has been formulated in terms of left QSDEs for technical reasons having to do with analysis of unbounded operators.

Generic cavity-QED scenario

For a (single-sided) cavity-QED system,

$$dU_t^* = \left\{\sqrt{2\kappa}\,(a\,dA_t^* - a^*dA_t) + \sqrt{\gamma_\parallel}\,(\sigma\,dB_t^* - \sigma^*dB_t) - (i\Delta_c+\kappa)a^*a\,dt - \left(i\Delta_a + \frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma\,dt + g_0(a^*\sigma - a\sigma^*)dt\right\}U_t^*,$$

$$dU_t = U_t\left\{-\sqrt{2\kappa}\,(a\,dA_t^* - a^*dA_t) - \sqrt{\gamma_\parallel}\,(\sigma\,dB_t^* - \sigma^*dB_t) + (i\Delta_c-\kappa)a^*a\,dt + \left(i\Delta_a - \frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma\,dt - g_0(a^*\sigma - a\sigma^*)dt\right\}.$$

Comparing this to the generic Hudson-Parthasarathy form, we have

$$K = (i\Delta_c-\kappa)a^*a + \left(i\Delta_a - \frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma - g_0(a^*\sigma - a\sigma^*),$$

$$L_1 = \sqrt{2\kappa}\,a^*, \quad L_2 = \sqrt{\gamma_\parallel}\,\sigma^*, \quad M_1 = -\sqrt{2\kappa}\,a, \quad M_2 = -\sqrt{\gamma_\parallel}\,\sigma, \quad N_{ij} = \delta_{ij}.$$

Checking the Hudson-Parthasarathy conditions,

$$
\begin{aligned}
K+K^* &= (i\Delta_c-\kappa)a^*a+\left(i\Delta_a-\frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma-g_0(a^*\sigma-a\sigma^*)-(i\Delta_c+\kappa)a^*a-\left(i\Delta_a+\frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma+g_0(a^*\sigma-a\sigma^*)\\
&= -2\kappa a^*a-\gamma_\parallel\sigma^*\sigma,\\
-\sum_{i=1}^{n}L_iL_i^* &= -\{2\kappa a^*a+\gamma_\parallel\sigma^*\sigma\},\\
-\sum_{j=1}^{n}N_{ij}L_j^* &= -L_i^* = M_i,
\end{aligned}
$$

and the condition on the N_{ij} is trivial.

Using our above result to intuit the master equation, we expect

$$
\begin{aligned}
K^*\rho_t+\rho_tK &= (-i\Delta_c-\kappa)a^*a\,\rho_t+\left(-i\Delta_a-\frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma\,\rho_t+g_0(a^*\sigma-a\sigma^*)\,\rho_t+\rho_t(i\Delta_c-\kappa)a^*a+\rho_t\left(i\Delta_a-\frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma-\rho_t g_0(a^*\sigma-a\sigma^*\\
&= -i[\{\Delta_c a^*a+\Delta_a\sigma^*\sigma+ig_0(a^*\sigma-a\sigma^*)\},\rho_t]-\kappa a^*a\rho_t-\kappa\rho_t a^*a-\frac{\gamma_\parallel}{2}\sigma^*\sigma\,\rho_t-\frac{\gamma_\parallel}{2}\rho_t\sigma^*\sigma,\\
\sum_{i=1}^{n}L_i^*\rho_tL_i &= 2\kappa a\rho_t a^*+\gamma_\parallel\sigma\rho_t\sigma^*,
\end{aligned}
$$

and the sum of these two terms gives us the master equation we expect.

Coherent input signal

To work with a coherent-state input we define the Weyl operator,

$$\psi(f) = W(f)\phi,$$

where $f\in L^2(R^+)$ specifies the amplitude and phase of the field at every time $t\in R^+$. All operators should now evolve according to

$$j_t(a) = W(f_t)^*U_t\,a\,U_t^*\,W(f_t),$$

where f_t denotes the function f truncated at time t. In general we have

$$
\begin{aligned}
dW(f_t) &= \left\{f(t)dA_t^*-f^*(t)dA_t-\frac{1}{2}|f(t)|^2\,dt\right\}W(f_t),\\
dW^*(f_t) &= W^*(f_t)\left\{f^*(t)dA_t-f(t)dA_t^*-\frac{1}{2}|f(t)|^2\,dt\right\},
\end{aligned}
$$

and defining

$$U_t^W = W^*(f_t)U_t,$$

the Itô rule gives us

$$dU_t^W = dW^*(f_t)U_t+W^*(f_t)dU_t+dW^*(f_t)dU_t.$$

With the general Hudson-Parthasarathy QSDE for dU_t, we thus have

$$
\begin{aligned}
dU_t^W &= W^*(f_t)\left\{f^*(t)dA_t^w-f(t)dA_t^{w*}-\frac{1}{2}|f(t)|dt\right\}U_t+W^*(f_t)U_t\left\{\sum_{i,j=1}^{n}(N_{ij}-\delta_{ij})d\Lambda_t^{ij}+\sum_{i=1}^{n}M_i(dA_t^i)^*+\sum_{i=1}^{n}L_i\,dA_t^i+K\,dt\right\}\\
&\quad+W^*(f_t)f^*(t)U_tM_wdt+W^*(f_t)U_tf^*(t)\sum_{j=1}^{n}(N_{wj}-\delta_{wj})dA_t^j\\
&= U_t^W\left\{\sum_{i,j=1}^{n}(N_{ij}-\delta_{ij})d\Lambda_t^{ij}+\sum_{i=1}^{n}(M_i-f(t)\delta_{iw})(dA_t^i)^*+\sum_{i=1}^{n}(L_i+f^*(t)N_{wi})dA_t^i+\left(K+f^*(t)M_w-\frac{1}{2}|f(t)|^2\right)dt\right\},
\end{aligned}
$$

where dA_t^w is the noise term associated with the displacement.

In the case of our cavity QED QSDE with $f=\alpha$ (constant), this would give us (dropping the W superscript on U_t^W)

$$
\begin{aligned}
dU_t &= U_t\Big\{\left(\sqrt{2\kappa}\,a^*+\alpha^*\right)dA_t-\left(\sqrt{2\kappa}\,a+\alpha\right)dA_t^*-\sqrt{\gamma_\parallel}\,(\sigma\,dB_t^*-\sigma^*dB_t)-\left(\sqrt{2\kappa}\,\alpha^*a+\frac{1}{2}|\alpha|^2\right)dt\\
&\quad+(i\Delta_c-\kappa)a^*a\,dt+\left(i\Delta_a-\frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma\,dt-g_0(a^*\sigma-a\sigma^*)dt\Big\}.
\end{aligned}
$$

We confirm that with the new

$$K = -\left(\sqrt{2\kappa}\,\alpha^* a + \frac{1}{2}|\alpha|^2\right) + (i\Delta_c - \kappa)a^* a + \left(i\Delta_a - \frac{1}{2}\gamma_\parallel\right)\sigma^*\sigma - g_0(a^*\sigma - a\sigma^*),$$

$$L_1 = \sqrt{2\kappa}\,a^* + \alpha^*, \quad L_2 = \sqrt{\gamma_\parallel}\,\sigma^*, \quad M_1 = -\left(\sqrt{2\kappa}\,a + \alpha\right), \quad M_2 = -\sqrt{\gamma_\parallel}\,\sigma, \quad N_{ij} = \delta_{ij},$$

we have

$$K + K^* = -\sqrt{2\kappa}\,(\alpha^* a + \alpha a^*) - 2\kappa a^* a - \gamma_\parallel \sigma^*\sigma - |\alpha|^2,$$

$$-\sum_{i=1}^{n} L_i L_i^* = -\left(\sqrt{2\kappa}\,a^* + \alpha^*\right)\left(\sqrt{2\kappa}\,a + \alpha\right) - \gamma_\parallel \sigma^*\sigma = -2\kappa a^* a - \sqrt{2\kappa}\,(\alpha a^* + \alpha^* a) - |\alpha|^2 - \gamma_\parallel \sigma^*\sigma,$$

which still satisfies the Hudson-Parthasarathy conditions. Looking at the implied master equation

$$\begin{aligned}
\dot{\rho}_t &= K^*\rho_t + \rho_t K + \sum_{i=1}^{n} L_i^*\rho_t L_i \\
&\to -i[\{\Delta_c a^* a + \Delta_a \sigma^*\sigma + ig_0(a^*\sigma - a\sigma^*)\}, \rho_t] - |\alpha|^2\rho_t - \sqrt{2\kappa}\,(\alpha a^*\rho_t + \alpha^*\rho_t a) - \kappa(\rho_t a^* a + a^* a\rho_t) - \frac{1}{2}\gamma_\parallel(\rho_t\sigma^*\sigma + \sigma^*\sigma\rho_t) \\
&\quad + \left(\sqrt{2\kappa}\,a + \alpha\right)\rho_t\left(\sqrt{2\kappa}\,a^* + \alpha^*\right) + \gamma_\parallel \sigma\rho_t\sigma^* \\
&= -i[\{\Delta_c a^* a + \Delta_a \sigma^*\sigma + ig_0(a^*\sigma - a\sigma^*)\}, \rho_t] + \kappa(2\kappa a\rho_t a^* - \rho_t a^* a - a^* a\rho_t) + \frac{\gamma_\parallel}{2}(\sigma\rho_t\sigma^* - \rho_t\sigma^*\sigma - \sigma^*\sigma\rho_t) \\
&\quad + \sqrt{2\kappa}\,(\alpha^* a\rho_t - \alpha^*\rho_t a - \alpha a^*\rho_t + \alpha\rho_t a^*) \\
&= -i\left[\left\{\Delta_c a^* a + \Delta_a \sigma^*\sigma + ig_0(a^*\sigma - a\sigma^*) - i\sqrt{2\kappa}\,(\alpha a^* - \alpha^* a)\right\}, \rho_t\right] \\
&\quad + \kappa(2\kappa a\rho_t a^* - \rho_t a^* a - a^* a\rho_t) + \frac{\gamma_\parallel}{2}(\sigma\rho_t\sigma^* - \rho_t\sigma^*\sigma - \sigma^*\sigma\rho_t),
\end{aligned}$$

we see that the effective change to the Hamiltonian is indeed the addition of a driving term $-i\sqrt{2\kappa}\,(\alpha a^* - \alpha^* a)$, as we have seen before.

Hopefully this calculation shows that in practice, it is much easier to use the $\triangleleft(1, \alpha, 0)$ method! But the above Weyl operator transformation is in fact the rigorous underpinning of the simplified method.

QSDE limit theorem of Bouten, van Handel and Silberfarb

Suppose we have a Hudson-Parthasarthy left QSDE whose coefficients contain a scaling factor k :

$$K^{(k)} = k^2 Y + kA + B, \quad L_i^{(k)} = kF_i + G_i, \quad N_{ij}^{(k)} = W_{ij}.$$

Suppose also that there exists a closed subspace $\mathbf{H}_0 \subset \mathbf{H}$ such that

1. $YP_0 = 0$;
2. There exists $\tilde{Y}$ such that $\tilde{Y}Y = Y\tilde{Y} = P_1$;
3. $F_j^\dagger P_0 = 0$ for all $1 \le j \le n$;
4. $P_0 A P_0 = 0$.

Here P_0 is an orthogonal projector onto $\mathbf{H}_0$ and $P_1 = 1 - P_0$ on $\mathbf{H}$. (The full statement of these 'structural conditions' is a bit more technical.) Then the theorem states that in the limit $k \to \infty$, the scaled QSDE limits to a QSDE on $\mathbf{H}_0$ with coefficients

$$K = P_0(B - A\tilde{Y}A)P_0, \quad L_i = P_0(G_i - A\tilde{Y}F_i)P_0,$$

$$M_i = -\sum_{j=1}^{n} P_0 W_{ij}(G_j^\dagger - F_j^\dagger \tilde{Y}A)P_0, \quad N_{ij} = \sum_{l=1}^{n} P_0 W_{il}(F_l^\dagger \tilde{Y}F_j + \delta_{lj})P_0.$$

This limit holds in the sense that

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left\| U_t^{(k)*}\psi - U_t^*\psi \right\| = 0 \quad \forall \psi \in \mathbf{H}_0 \otimes \mathbf{F}, \quad T < \infty,$$

where $U_t^{(k)}$ is the solution of the Hudson-Parthasarathy Equation (the Heisenberg-picture propagator) in the original model with scaling parameter k, and U_t is the solution in the limit QSDE. Thus $U_t^*\psi$ is the solution of the Schrödinger Equation on the joint state-space of the system and its input-output fields, and the limit stated above amounts to strong uniform convergence on compact time intervals. The proof is rather technical and relies on a version of the Trotter-Kato theorm for QSDEs – see Bouten, Silberfarb and Van Handel (2008) cited above.

The 'small volume' limit

We are interested in using the above theorem to obtain component model abstractions in limits that are relevant to nanophotonic device physics. One such limit is the small volume limit in which g and κ both naturally become large. In particular we may be interested in scenarios in which $g, \kappa \to \infty$ such that g/κ stays fixed, or perhaps g^2/κ stays fixed. We have previously noted that g^2 tends to scale inversely with the volume of a resonator, and for a Fabry-Perot type cavity one can note that κ will tend to increase as the cavity volume decreases simply because the circulating light will bounce off of the end mirrors more often. For monolithic or photonic-crystal defect resonators, κ also increases as the volume decreases because of fundamental limits on the confinement of light within structures of sub-wavelength dimension. The other parameters appearing in these models, which retain their original values in the limit model, are things like atomic decay rates and input field powers. As briefly discussed in the above-referenced paper on switch models, the regime with $(g \sim \kappa) \gg \gamma$ seems accessible with current systems of interest for nanophotonic cavity QED.

Four-state, three-cavity model for the SR flip-flop switch

We now attempt to derive an SR flip-flop switch model using a four-level atom coupled to three cavity modes. The QSDE terms are as follows:

$$K = -k_1^2\kappa_a a^* a - k_1^2 g_a(a^*\sigma_{ge} - a\sigma_{ge}^*) - k_2^2\kappa_b b^* b - k_2^2\kappa_c c^* c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*),$$

$$L_1 = k_1\sqrt{\kappa_a}\,a^*, \quad M_1 = -k_1\sqrt{\kappa_a}\,a,$$

$$L_2 = k_1\sqrt{\kappa_a}\,a^*, \quad M_2 = -k_1\sqrt{\kappa_a}\,a,$$

$$L_3 = k_2\sqrt{2\kappa_b}\,b^*, \quad M_3 = -k_2\sqrt{2\kappa_b}\,b,$$

$$L_4 = k_2\sqrt{2\kappa_c}\,c^*, \quad M_4 = -k_2\sqrt{2\kappa_c}\,c,$$

$$N_{ij} = \delta_{ij}.$$

The kind of situation we have in mind here involves a multilevel 'atom' with ground states $\{|h\rangle, |g\rangle\}$ and excited states $\{|e\rangle, |r\rangle\}$. Cavity mode a connects $|g\rangle \leftrightarrow |e\rangle$ only, cavity mode b connects $|g\rangle \leftrightarrow |r\rangle$ (set) and $|h\rangle \leftrightarrow |e\rangle$, and cavity mode c connects $|h\rangle \leftrightarrow |r\rangle$ (reset). (Note that this QSDE corresponds to the "Simplified four-state relay model" in the accompanying PowerPoint slides, but the paper referenced above assumes a different atomic level structure that can be used when the control field and routed field for the switch/relay have very different frequencies.)

We will first take the limit $k_1 \to \infty$ and then $k_2 \to \infty$. For the first elimination,

$$K = k^2 Y + kA + B, \quad L_i = kF_i + G_i, \quad N_{ij} = W_{ij},$$

$$Y = -\kappa_a a^* a - g_a(a^*\sigma_{ge} - a\sigma_{ge}^*), \quad A = 0, \quad B = -k_2^2\kappa_b b^* b - k_2^2\kappa_c c^* c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*),$$

$$F = \begin{pmatrix} \sqrt{\kappa_a}\,a^* \\ \sqrt{\kappa_a}\,a^* \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ k_2\sqrt{2\kappa_b}\,b^* \\ k_2\sqrt{2\kappa_c}\,c^* \end{pmatrix}, \quad W_{ij} = \delta_{ij}.$$

We choose $H_0 = \mathrm{span}\{|g\,0_a\,n_b\,n_c\rangle, |h\,0_a\,n_b\,n_c\rangle, |r\,0_a\,n_b\,n_c\rangle\}$, which clearly lies within the kernel of Y. We define $\tilde{Y}$ by its action on an orthonormal basis of states. We begin with basis states of the form $\{|g\,0_a\,n_b\,n_c\rangle, |h\,0_a\,n_b\,n_c\rangle, |r\,0_a\,n_b\,n_c\rangle\}$, for which

$$Y\{|g\,0_a\,n_b\,n_c\rangle, |h\,0_a\,n_b\,n_c\rangle, |r\,0_a\,n_b\,n_c\rangle\} = 0,$$

$$\tilde{Y}\{|g\,0_a\,n_b\,n_c\rangle, |h\,0_a\,n_b\,n_c\rangle, |r\,0_a\,n_b\,n_c\rangle\} = 0.$$

This ensures that $Y\tilde{Y}P_0 = \tilde{Y}YP_0 = 0$. We next note that for $n \geq 1$,

$$Y|g n_a n_b n_c\rangle = -\kappa_a n_a |g n_a n_b n_c\rangle + g_a\sqrt{n_a}\,|e(n_a-1)_a n_b n_c\rangle, \quad n_a \geq 1,$$
$$Y|e(n-1)_a n_b n_c\rangle = -\kappa_a(n_a-1)|e(n_a-1)_a n_b n_c\rangle - g_a\sqrt{n_a}\,|g n_a n_b n_c\rangle, \quad n_a \geq 1,$$
$$Y|h n_a n_b n_c\rangle = -\kappa_a n_a |h n_a n_b n_c\rangle, \quad n_a \geq 1,$$
$$Y|r n_a n_b n_c\rangle = -\kappa_a n_a |r n_a n_b n_c\rangle, \quad n_a \geq 1.$$

We can thus define

$$\tilde{Y}|r n_a n_b n_c\rangle = -\frac{1}{\kappa_a n_a}|r n_a n_b n_c\rangle, \quad n_a \geq 1,$$
$$\tilde{Y}|h n_a n_b n_c\rangle = -\frac{1}{\kappa_a n_a}|h n_a n_b n_c\rangle, \quad n_a \geq 1,$$
$$\tilde{Y}|e(n-1)_a n_b n_c\rangle = \frac{g_a\sqrt{n_a}}{\kappa_a^2 n_a(n_a-1) - g_a^2 n_a}|g n_a n_b n_c\rangle - \frac{\kappa_a n_a}{\kappa_a^2 n_a(n_a-1) - g_a^2 n_a}|e(n_a-1)_a n_b n_c\rangle, \quad n_a \geq 1,$$
$$\tilde{Y}|g n_a n_b n_c\rangle = -\frac{\kappa_a(n_a-1)}{\kappa_a^2 n_a(n_a-1) - g_a^2 n_a}|g n_a n_b n_c\rangle + \frac{g_a\sqrt{n_a}}{\kappa_a^2 n_a(n_a-1) - g_a^2 n_a}|e(n_a-1)_a n_b n_c\rangle, \quad n_a \geq 1,$$

and thus satisfy $Y\tilde{Y} = \tilde{Y}Y = P_1$. Checking the remaining structural conditions, we require

$$F_i^* P_0 = 0,$$

which is evident by inspection, and

$$P_0 A P_0 = 0,$$

which is likewise trivial. Moving on to compute the limit coefficients

$$\check{K} = P_0(B - A\tilde{Y}A)P_0, \quad \check{L}_i = P_0(G_i - A\tilde{Y}F_i)P_0,$$
$$\check{M}_i = -\sum_{j=1}^{n} P_0 W_{ij}(G_j^* - F_j^*\tilde{Y}A)P_0, \quad \check{N}_{ij} = \sum_{l=1}^{n} P_0 W_{il}(F_l^*\tilde{Y}F_j + \delta_{lj})P_0,$$

we first obtain

$$\begin{aligned}
P_0BP_0 &= \{-k_2^2\kappa_b b^*b - k_2^2\kappa_c c^*c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)\}P_0 \\
&= -k_2^2\kappa_b b^*b - k_2^2\kappa_c c^*c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_b P_0(b^*\sigma_{he} - b\sigma_{he}^*)P_0 - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*) \\
&= -k_2^2\kappa_b b^*b - k_2^2\kappa_c c^*c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*), \\
\check{K} &= -k_2^2\kappa_b b^*b - k_2^2\kappa_c c^*c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*).
\end{aligned}$$

Next

$$\check{L}_1 = \check{L}_2 = 0, \quad \check{L}_3 = k_2\sqrt{2\kappa_b}\,b^*, \quad \check{L}_4 = k_2\sqrt{2\kappa_c}\,c^*,$$

and

$$\check{M}_1 = \check{M}_2 = 0, \quad \check{M}_3 = -k_2\sqrt{2\kappa_b}\,b, \quad \check{M}_3 = -k_2\sqrt{2\kappa_c}\,c.$$

To compute the scattering matrix we first note

$$\begin{aligned}
a^*P_0 &= |g1_a\rangle\langle g0_a| \otimes 1_b \otimes 1_c + |h1_a\rangle\langle h0_a| \otimes 1_b \otimes 1_c + |r1_a\rangle\langle r0_a| \otimes 1_b \otimes 1_c, \\
\tilde{Y}a^*P_0 &= -\frac{1}{g_a}|e0_a\rangle\langle g0_a| \otimes 1_b \otimes 1_c - \frac{1}{\kappa_a}|h1_a\rangle\langle h0_a| \otimes 1_b \otimes 1_c - \frac{1}{\kappa_a}|r1_a\rangle\langle r0_a| \otimes 1_b \otimes 1_c, \\
a\tilde{Y}a^*P_0 &= -\frac{1}{\kappa_a}|h0_a\rangle\langle h0_a| \otimes 1_b \otimes 1_c - \frac{1}{\kappa_a}|r0_a\rangle\langle r0_a| \otimes 1_b \otimes 1_c, \\
\kappa_a P_0 a\tilde{Y}a^*P_0 &= -\{\Pi_{h0_a} + \Pi_r 0_a\} \otimes 1_b \otimes 1_c.
\end{aligned}$$

$$\check{N}_{11} = P_0(F_1^*\tilde{Y}F_1 + 1)P_0 = 1 + \kappa_a P_0 a\tilde{Y}a^*P_0 = \Pi_g,$$
$$\check{N}_{12} = P_0F_1^*\tilde{Y}F_2P_0 = \kappa_a P_0 a\tilde{Y}a^*P_0 = -\Pi_{hr},$$
$$\check{N}_{13} = P_0(F_1^*\tilde{Y}F_3)P_0 = 0,$$
$$\check{N}_{14} = P_0(F_1^*\tilde{Y}F_4)P_0 = 0,$$
$$\check{N}_{21} = P_0(F_2^*\tilde{Y}F_1)P_0 = \kappa_a P_0 a\tilde{Y}a^*P_0 = -\Pi_{hr},$$
$$\check{N}_{22} = P_0(F_2^*\tilde{Y}F_2 + 1)P_0 = 1 + \kappa_a P_0 a\tilde{Y}a^*P_0 = \Pi_g$$
$$\check{N}_{23} = P_0(F_2^*\tilde{Y}F_3)P_0 = 0,$$

$$\breve{N}_{24} = P_0(F_2^*\tilde{Y}F_4)P_0 = 0,$$

$$\breve{N}_{31} = P_0(F_3^*\tilde{Y}F_1)P_0 = 0,$$

$$\breve{N}_{32} = P_0(F_3^*\tilde{Y}F_2 + 1)P_0 = 0,$$

$$\breve{N}_{33} = P_0(F_3^*\tilde{Y}F_3 + 1)P_0 = 1,$$

$$\breve{N}_{34} = P_0(F_4^*\tilde{Y}F_4)P_0 = 0,$$

$$\breve{N}_{41} = P_0(F_4^*\tilde{Y}F_1)P_0 = 0,$$

$$\breve{N}_{42} = P_0(F_4^*\tilde{Y}F_2)P_0 = 0,$$

$$\breve{N}_{43} = P_0(F_4^*\tilde{Y}F_3)P_0 = 0,$$

$$\breve{N}_{44} = P_0(F_4^*\tilde{Y}F_4 + 1)P_0 = 1.$$

Collecting together the results, we have

$$K = -k_2^2\kappa_b b^* b - k_2^2\kappa_c c^* c - k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*),$$

$$L = \begin{pmatrix} 0 \\ 0 \\ k_2\sqrt{2\kappa_b}\, b^* \\ k_2\sqrt{2\kappa_c}\, c^* \end{pmatrix}, \quad M = \begin{pmatrix} 0 \\ 0 \\ -k_2\sqrt{2\kappa_b}\, b \\ -k_2\sqrt{2\kappa_c}\, c \end{pmatrix}, \quad N = \begin{pmatrix} \Pi_g & -\Pi_{hr} & 0 & 0 \\ -\Pi_{hr} & \Pi_g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Checking the H-P conditions,

$$K + K^* = -2k_2^2\kappa_b b^* b - 2k_2^2\kappa_c c^* c,$$

$$-\sum_{i=1}^{4} L_i L_i^* = -k_2^2 2\kappa_b b^* b - k_2^2 2\kappa_c c^* c,$$

and

$$-NL^* = \begin{pmatrix} -\Pi_g & \Pi_{hr} & 0 & 0 \\ \Pi_{hr} & -\Pi_g & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ k_2\sqrt{2\kappa_b}\, b \\ k_2\sqrt{2\kappa_c}\, c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -k_2\sqrt{2\kappa_b}\, b \\ -k_2\sqrt{2\kappa_c}\, c \end{pmatrix} = M,$$

$$NN = \begin{pmatrix} \Pi_g & -\Pi_{hr} & 0 & 0 \\ -\Pi_{hr} & \Pi_g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Pi_g & -\Pi_{hr} & 0 & 0 \\ -\Pi_{hr} & \Pi_g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so everything looks okay after the first elimination.

For the second elimination we have

$$Y = -\kappa_b b^* b - \kappa_c c^* c, \quad A = -g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - g_c(c^*\sigma_{hr} - c\sigma_{hr}^*), \quad B = 0,$$

$$F = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2\kappa_b}\, b^* \\ \sqrt{2\kappa_c}\, c^* \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} \Pi_g & -\Pi_{hr} & 0 & 0 \\ -\Pi_{hr} & \Pi_g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We choose $H_0 = \text{span}\{|g0_b0_c\rangle, |h0_b0_c\rangle, |r0_b0_c\rangle\}$, which clearly lies in the kernel of Y. We define

$$\tilde{Y}|g0_b0_c\rangle = \tilde{Y}|h0_b0_c\rangle = \tilde{Y}|r0_b0_c\rangle = 0,$$

and

$$\tilde{Y}|g\,n_b\,n_c\rangle = -\frac{1}{\kappa_b n_b + \kappa_c n_c}|g\,n_b\,n_c\rangle, \quad n_b + n_c \geq 1,$$
$$\tilde{Y}|h\,n_b\,n_c\rangle = -\frac{1}{\kappa_b n_b + \kappa_c n_c}|h\,n_b\,n_c\rangle, \quad n_b + n_c \geq 1,$$
$$\tilde{Y}|r\,n_b\,n_c\rangle = -\frac{1}{\kappa_b n_b + \kappa_c n_c}|r\,n_b\,n_c\rangle, \quad n_b + n_c \geq 1,$$

which achieves the desired condition $Y\tilde{Y} = \tilde{Y}Y = P_1$. Checking the remaining structural conditions,

$$F_1^* P_0 = F_2^* P_0 = 0, \quad F_3^* P_0 = \sqrt{2\kappa_b}\, b P_0 = 0, \quad F_4^* P_0 = \sqrt{2\kappa_c}\, c P_0 = 0,$$

and

$$\begin{aligned} AP_0 &= \{-g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) - g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)\}(\Pi_g + \Pi_h + \Pi_r) \otimes |0_b 0_c\rangle\langle 0_b 0_c| \\ &= -g_b\sigma_{gr} \otimes |1_b 0_c\rangle\langle 0_b 0_c| - g_c\sigma_{hr} \otimes |0_b 1_c\rangle\langle 0_b 0_c|, \\ P_0 A P_0 &= 0. \end{aligned}$$

Hence we can move on to computing the limit coefficients,

$$\check{K} = P_0(B - A\tilde{Y}A)P_0, \quad \check{L}_i = P_0(G_i - A\tilde{Y}F_i)P_0,$$
$$\check{M}_i = -\sum_{j=1}^{n} P_0 W_{ij}(G_j^* - F_j^*\tilde{Y}A)P_0, \quad \check{N}_{ij} = \sum_{l=1}^{n} P_0 W_{il}(F_l^*\tilde{Y}F_j + \delta_{lj})P_0.$$

Starting with the first, we have

$$\begin{aligned} AP_0 &= -g_b\sigma_{gr} \otimes |1_b 0_c\rangle\langle 0_b 0_c| - g_c\sigma_{hr} \otimes |0_b 1_c\rangle\langle 0_b 0_c|, \\ \tilde{Y}AP_0 &= \frac{g_b}{\kappa_b}\sigma_{gr} \otimes |1_b 0_c\rangle\langle 0_b 0_c| + \frac{g_c}{\kappa_c}\sigma_{hr} \otimes |0_b 1_c\rangle\langle 0_b 0_c|, \\ A\tilde{Y}AP_0 &= \frac{g_b^2}{\kappa_b}\Pi_r|0_b 0_c\rangle\langle 0_b 0_c| + \frac{g_c^2}{\kappa_c}\Pi_r|0_b 0_c\rangle\langle 0_b 0_c|, \\ \check{K} &= -P_0 A\tilde{Y}AP_0 = -\left(\frac{g_b^2}{\kappa_b} + \frac{g_c^2}{\kappa_c}\right)\Pi_r. \end{aligned}$$

Next

$$\check{L}_1 = \check{L}_2 = 0,$$

$$\begin{aligned} F_3P_0 &= \sqrt{2\kappa_b}\,(\Pi_g + \Pi_h + \Pi_r) \otimes |1_b 0_c\rangle\langle 0_b 0_c|, \\ \tilde{Y}F_3P_0 &= -\sqrt{\frac{2}{\kappa_b}}\,(\Pi_g + \Pi_h + \Pi_r) \otimes |1_b 0_c\rangle\langle 0_b 0_c|, \\ A\tilde{Y}F_3P_0 &= -\sqrt{\frac{2}{\kappa_b}}\left\{g_b\sigma_{rg} \otimes |0_b 0_c\rangle\langle 0_b 0_c| - \sqrt{2}\,g_b\sigma_{gr} \otimes |2_b 0_c\rangle\langle 0_b 0_c| - g_c\sigma_{hr} \otimes |1_b 1_c\rangle\langle 0_b 0_c|\right\}, \\ \check{L}_3 &= -P_0 A\tilde{Y}F_3P_0 = \sqrt{\frac{2g_b^2}{\kappa_b}}\,\sigma_{rg}, \\ F_4P_0 &= \sqrt{2\kappa_c}\,(\Pi_g + \Pi_h + \Pi_r) \otimes |0_b 1_c\rangle\langle 0_b 0_c|, \\ \tilde{Y}F_4P_0 &= -\sqrt{\frac{2}{\kappa_c}}\,(\Pi_g + \Pi_h + \Pi_r) \otimes |0_b 1_c\rangle\langle 0_b 0_c|, \\ A\tilde{Y}F_4P_0 &= -\sqrt{\frac{2}{\kappa_c}}\left\{g_c\sigma_{rh} \otimes |0_b 0_c\rangle\langle 0_b 0_c| - g_b\sigma_{gr} \otimes |1_b 1_c\rangle\langle 0_b 0_c| - \sqrt{2}\,g_c\sigma_{hr} \otimes |0_b 2_c\rangle\langle 0_b 0_c|\right\}, \\ \check{L}_4 &= -P_0 A\tilde{Y}F_4P_0 = \sqrt{\frac{2g_c^2}{\kappa_c}}\,\sigma_{rh}. \end{aligned}$$

Next

$$\check{M}_1 = P_0\Pi_g F_1^*\tilde{Y}AP_0 - P_0\Pi_{hr}F_2^*\tilde{Y}AP_0 = 0,$$
$$\check{M}_2 = -P_0\Pi_{hr}F_1^*\tilde{Y}AP_0 + P_0\Pi_g F_2^*\tilde{Y}AP_0 = 0,$$

$$\check{M}_3 = -P_0(-F_3^*\tilde{Y}A)P_0 = P_0\sqrt{2\kappa_b}\,b\left\{\frac{g_b}{\kappa_b}\sigma_{gr} \otimes |1_b 0_c\rangle\langle 0_b 0_c| + \frac{g_c}{\kappa_c}\sigma_{hr} \otimes |0_b 1_c\rangle\langle 0_b 0_c|\right\} = \sqrt{\frac{2g_b^2}{\kappa_b}}\,\sigma_{gr},$$

$$\check{M}_4 = -P_0(-F_4^*\tilde{Y}A)P_0 = P_0\sqrt{2\kappa_c}\,c\left\{\frac{g_b}{\kappa_b}\sigma_{gr} \otimes |1_b 0_c\rangle\langle 0_b 0_c| + \frac{g_c}{\kappa_c}\sigma_{hr} \otimes |0_b 1_c\rangle\langle 0_b 0_c|\right\} = \sqrt{\frac{2g_c^2}{\kappa_c}}\,\sigma_{hr}.$$

Finally

$$\breve{N}_{11} = P_0 W_{11}(F_1^* \tilde{Y} F_1 + \delta_{11}) P_0 + P_0 W_{12}(F_2^* \tilde{Y} F_1 + \delta_{21}) P_0 = P_0 W_{11} P_0 = \Pi_g,$$

$$\breve{N}_{12} = P_0 W_{11}(F_1^* \tilde{Y} F_2 + \delta_{12}) P_0 + P_0 W_{12}(F_2^* \tilde{Y} F_2 + \delta_{22}) P_0 = P_0 W_{12} P_0 = -\Pi_{hr},$$

$$\breve{N}_{21} = P_0 W_{21}(F_1^* \tilde{Y} F_1 + \delta_{11}) P_0 + P_0 W_{22}(F_2^* \tilde{Y} F_1 + \delta_{21}) P_0 = P_0 W_{21} P_0 = -\Pi_{hr},$$

$$\breve{N}_{22} = P_0 W_{21}(F_1^* \tilde{Y} F_2 + \delta_{12}) P_0 + P_0 W_{22}(F_2^* \tilde{Y} F_2 + \delta_{22}) P_0 = P_0 W_{22} P_0 = \Pi_g,$$

$$\breve{N}_{33} = P_0 W_{33}(F_3^* \tilde{Y} F_3 + \delta_{33}) P_0 = 1 + P_0 F_3^* \tilde{Y} F_3 P_0 = 1 - 2(\Pi_g + \Pi_h + \Pi_r) \otimes |0_b 0_c\rangle\langle 0_b 0_c| = -1,$$

$$\breve{N}_{34} = P_0 W_{33}(F_3^* \tilde{Y} F_4 + \delta_{34}) P_0 = P_0 F_3^* \tilde{Y} F_4 P_0 = P_0 2\sqrt{\kappa_b \kappa_c}\, b(\Pi_g + \Pi_h + \Pi_r) \otimes |0_b 1_c\rangle\langle 0_b 0_c| = 0,$$

$$\breve{N}_{43} = P_0 W_{44}(F_4^* \tilde{Y} F_3 + \delta_{43}) P_0 = P_0 F_4^* \tilde{Y} F_3 P_0 = P_0 2\sqrt{\kappa_b \kappa_c}\, c(\Pi_g + \Pi_h + \Pi_r) \otimes |1_b 0_c\rangle\langle 0_b 0_c| = 0,$$

$$\breve{N}_{44} = P_0 W_{44}(F_4^* \tilde{Y} F_4 + \delta_{44}) P_0 = 1 + P_0 F_4^* \tilde{Y} F_4 P_0 = 1 - 2(\Pi_g + \Pi_h + \Pi_r) \otimes |0_b 0_c\rangle\langle 0_b 0_c| = -1.$$

Collecting the results, and setting

$$\frac{2g_b^2}{\kappa_b} = \frac{2g_c^2}{\kappa_c} \equiv \gamma,$$

$$K = -\gamma \Pi_r,$$

$$L = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\gamma}\,\sigma_{rg} \\ \sqrt{\gamma}\,\sigma_{rh} \end{pmatrix}, \quad M = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\gamma}\,\sigma_{gr} \\ \sqrt{\gamma}\,\sigma_{hr} \end{pmatrix}, \quad N = \begin{pmatrix} \Pi_g & -\Pi_{hr} & 0 & 0 \\ -\Pi_{hr} & \Pi_g & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now we are finally ready to take the limit $\gamma \to \infty$. We set

$$Y = -\gamma \Pi_r, \quad A = B = 0,$$

$$F = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\gamma}\,\sigma_{rg} \\ \sqrt{\gamma}\,\sigma_{rh} \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} \Pi_g & -\Pi_{hr} & 0 & 0 \\ -\Pi_{hr} & \Pi_g & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then $H_0 = \mathrm{span}\{|g\rangle, |h\rangle\}$ and we set

$$\tilde{Y}|g\rangle = \tilde{Y}|h\rangle = 0, \quad \tilde{Y}|r\rangle = -\gamma^{-1}|r\rangle.$$

Then

$$\breve{K} = 0,$$

$$\breve{L}_1 = \breve{L}_2 = \breve{L}_3 = \breve{L}_4 = 0,$$

$$\breve{M}_1 = \breve{M}_2 = \breve{M}_3 = \breve{M}_4 = 0,$$

$$\breve{N}_{11} = P_0 W_{11}(F_1^* \tilde{Y} F_1 + \delta_{11}) P_0 + P_0 W_{12}(F_2^* \tilde{Y} F_1 + \delta_{21}) P_0 = \Pi_g$$

$$\breve{N}_{12} = P_0 W_{11}(F_1^* \tilde{Y} F_2 + \delta_{12}) P_0 + P_0 W_{12}(F_2^* \tilde{Y} F_2 + \delta_{22}) P_0 = -\Pi_h$$

$$\breve{N}_{21} = P_0 W_{21}(F_1^* \tilde{Y} F_1 + \delta_{11}) P_0 + P_0 W_{22}(F_2^* \tilde{Y} F_1 + \delta_{21}) P_0 = -\Pi_h$$

$$\breve{N}_{22} = P_0 W_{21}(F_1^* \tilde{Y} F_2 + \delta_{12}) P_0 + P_0 W_{22}(F_2^* \tilde{Y} F_2 + \delta_{22}) P_0 = \Pi_g$$

$$\breve{N}_{33} = P_0 W_{33}(F_3^* \tilde{Y} F_3 + \delta_{33}) P_0 = 1 - P_0 F_3^* \tilde{Y} F_3 P_0 = 1 - \Pi_g = \Pi_h,$$

$$\breve{N}_{34} = P_0 W_{33}(F_3^* \tilde{Y} F_4 + \delta_{34}) P_0 = -P_0 F_3^* \tilde{Y} F_4 P_0 = -\sigma_{gr}\sigma_{rh} = -\sigma_{gh},$$

$$\breve{N}_{43} = P_0 W_{44}(F_4^* \tilde{Y} F_3 + \delta_{43}) P_0 = -P_0 F_4^* \tilde{Y} F_3 P_0 = -\sigma_{hr}\sigma_{rg} = -\sigma_{hg},$$

$$\breve{N}_{44} = P_0 W_{44}(F_4^* \tilde{Y} F_4 + \delta_{44}) P_0 = 1 - P_0 F_4^* \tilde{Y} F_4 P_0 = 1 - \Pi_h = \Pi_g.$$

Collecting the results together,

$$K = 0, \quad L = 0, \quad M = 0, \quad N = \begin{pmatrix} \Pi_g & -\Pi_h & 0 & 0 \\ -\Pi_h & \Pi_g & 0 & 0 \\ 0 & 0 & \Pi_h & -\sigma_{gh} \\ 0 & 0 & -\sigma_{hg} & \Pi_g \end{pmatrix}.$$

We should keep in mind that these are H-P coefficients for a left QSDE model; generally speaking we convert to the (S_r, L_r, H_r) triple for a right QSDE model via

$$S_r = N^\dagger, \quad L_r = L^\#, \quad H_r = \mathrm{Im}[K],$$

where for a vector or matrix of operators or complex numbers $A = [a_{ij}]$, we define $A^\# \equiv [a_{ij}^*]$ where a_{ij}^* is the complex conjugate of a number or the hermitian conjugate of an operator, and $A^\dagger \equiv [a_{ji}^*] = (A^\#)^T$.

Model with coherent signals

Finally we add displacements to modes 1, 3 and 4 to obtain a model with known, coherent signals. With the general rules

$$M_i \to M_i - f_w \delta_{iw}, \quad L_i \to L_i + f_w^* N_{wi}, \quad K \to K + f_w^* M_w - \frac{1}{2}|f_w|^2,$$

and $f_1 = \beta$, $f_3 = \alpha_s$, $f_4 = \alpha_r$,

$$K = -\frac{1}{2}|\beta|^2 - \frac{1}{2}|\alpha_s|^2 - \frac{1}{2}|\alpha_r|^2,$$

$$L = \begin{pmatrix} \beta^* \Pi_g \\ -\beta^* \Pi_h \\ \alpha_s^* \Pi_h - \alpha_r^* \sigma_{hg} \\ -\alpha_s^* \sigma_{gh} + \alpha_r^* \Pi_g \end{pmatrix}, \quad M = \begin{pmatrix} -\beta \\ 0 \\ -\alpha_s \\ -\alpha_r \end{pmatrix}, \quad N = \begin{pmatrix} \Pi_g & -\Pi_h & 0 & 0 \\ -\Pi_h & \Pi_g & 0 & 0 \\ 0 & 0 & \Pi_h & -\sigma_{gh} \\ 0 & 0 & -\sigma_{hg} & \Pi_g \end{pmatrix}.$$

The master equation then reads

$$\begin{aligned}
\dot{\rho}_t &= -|\beta|^2 \rho_t - |\alpha_s|^2 \rho_t - |\alpha_r|^2 \rho_t + |\beta|^2 \{\Pi_g \rho_t \Pi_g + \Pi_h \rho_t \Pi_h\} + (\alpha_s \Pi_h - \alpha_r \sigma_{gh}) \rho_t (\alpha_s^* \Pi_h - \alpha_r^* \sigma_{gh}^*) + (\alpha_s \sigma_{hg} - \alpha_r \Pi_g) \rho_t (\alpha_s^* \sigma_{hg}^* - \alpha_r^* \Pi_g) \\
&= -|\beta|^2 \{\Pi_g \rho_t \Pi_h + \Pi_h \rho_t \Pi_g\} - |\alpha_s|^2 \rho_t - |\alpha_r|^2 \rho_t + (\alpha_s \Pi_h - \alpha_r \sigma_{gh}) \rho_t (\alpha_s^* \Pi_h - \alpha_r^* \sigma_{gh}^*) + (\alpha_s \sigma_{hg} - \alpha_r \Pi_g) \rho_t (\alpha_s^* \sigma_{hg}^* - \alpha_r^* \Pi_g) \\
&= \left(|\beta|^2 + \frac{|\alpha_s|^2}{2} + \frac{|\alpha_r|^2}{2}\right) \{Z \rho_t Z - \rho_t\} + \frac{|\alpha_s|^2}{2} \{2\sigma_{hg} \rho_t \sigma_{hg}^* - \sigma_{hg}^* \sigma_{hg} \rho_t - \rho_t \sigma_{hg}^* \sigma_{hg}\} + \frac{|\alpha_r|^2}{2} \{2\sigma_{gh} \rho_t \sigma_{gh}^* - \sigma_{gh}^* \sigma_{gh} \rho_t - \rho_t \sigma_{gh}^* \sigma_{gh}\} \\
&\quad - (\alpha_s^* \alpha_r \sigma_{gh}) \rho_t \Pi_h - \Pi_h \rho_t (\alpha_s^* \alpha_r \sigma_{gh})^* - (\alpha_s \alpha_r^* \sigma_{hg}) \rho_t \Pi_g - \Pi_g \rho_t (\alpha_s \alpha_r^* \sigma_{hg})^*,
\end{aligned}$$

where $Z \equiv \Pi_g - \Pi_h$.

Pre-limit master equation with displacements and spontaneous emission

For use in numerical simulations we apply displacements to the pre-limit model as well, obtaining

$$\begin{aligned}
K &= -k_1^2 \kappa_a a^* a - k_2^2 \kappa_b b^* b - k_2^2 \kappa_c c^* c - k_1^2 g_a (a^* \sigma_{ge} - a \sigma_{ge}^*) - k_2 g_b (b^* \sigma_{gr} - b \sigma_{gr}^*) - k_2 g_b (b^* \sigma_{he} - b \sigma_{he}^*) - k_2 g_c (c^* \sigma_{hr} - c \sigma_{hr}^*) \\
&\quad - \beta^* k_1 \sqrt{\kappa_a}\, a - \alpha_s^* k_2 \sqrt{2\kappa_b}\, b - \alpha_r^* k_3 \sqrt{2\kappa_c}\, c - \frac{1}{2}|\beta|^2 - \frac{1}{2}|\alpha_s|^2 - \frac{1}{2}|\alpha_r|^2,
\end{aligned}$$

$$L_1 = k_1 \sqrt{\kappa_a}\, a^* + \beta^*, \quad M_1 = -k_1 \sqrt{\kappa_a}\, a - \beta,$$

$$L_2 = k_1 \sqrt{\kappa_a}\, a^*, \quad M_2 = -k_1 \sqrt{\kappa_a}\, a,$$

$$L_3 = k_2 \sqrt{2\kappa_b}\, b^* + \alpha_s^*, \quad M_3 = -k_2 \sqrt{2\kappa_b}\, b - \alpha_s,$$

$$L_4 = k_2 \sqrt{2\kappa_c}\, c^* + \alpha_r^*, \quad M_4 = -k_2 \sqrt{2\kappa_c}\, c - \alpha_r,$$

$$N_{ij} = \delta_{ij}.$$

The corresponding master equation is then

$$\dot{\rho}_t = -i[i\{k_1^2 g_a(a^*\sigma_{ge} - a\sigma_{ge}^*) + k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) + k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) + k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)\}, \rho_t]$$
$$+ \left(-\beta k_1\sqrt{\kappa_a}\,a^* - \alpha_s k_2\sqrt{2\kappa_b}\,b^* - \alpha_r k_3\sqrt{2\kappa_c}\,c^*\right)\rho_t + \rho_t\left(-\beta^* k_1\sqrt{\kappa_a}\,a - \alpha_s^* k_2\sqrt{2\kappa_b}\,b - \alpha_r^* k_3\sqrt{2\kappa_c}\,c\right)$$
$$- |\beta|^2\rho_t - |\alpha_s|^2\rho_t - |\alpha_r|^2\rho_t$$
$$+ k_1^2\kappa_a(-a^*a\rho_t - \rho_t a^*a) + k_2^2\kappa_b(-b^*b\rho_t - \rho_t b^*b) + k_2^2\kappa_c(-c^*c\rho_t - \rho_t c^*c)$$
$$+ (k_1\sqrt{\kappa_a}\,a + \beta)\rho_t(k_1\sqrt{\kappa_a}\,a^* + \beta^*) + (k_1\sqrt{\kappa_a}\,a)\rho_t(k_1\sqrt{\kappa_a}\,a^*) + \left(k_2\sqrt{2\kappa_b}\,b + \alpha_s\right)\rho_t\left(k_2\sqrt{2\kappa_b}\,b^* + \alpha_s^*\right)$$
$$+ \left(k_2\sqrt{2\kappa_c}\,c + \alpha_r\right)\rho_t\left(k_2\sqrt{2\kappa_c}\,c^* + \alpha_r^*\right).$$

Simplifying,

$$\dot{\rho}_t = -i[i\{k_1^2 g_a(a^*\sigma_{ge} - a\sigma_{ge}^*) + k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) + k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) + k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)\}, \rho_t]$$
$$+ \left(-\beta k_1\sqrt{\kappa_a}\,a^* - \alpha_s k_2\sqrt{2\kappa_b}\,b^* - \alpha_r k_3\sqrt{2\kappa_c}\,c^*\right)\rho_t + \rho_t\left(-\beta^* k_1\sqrt{\kappa_a}\,a - \alpha_s^* k_2\sqrt{2\kappa_b}\,b - \alpha_r^* k_3\sqrt{2\kappa_c}\,c\right)$$
$$- |\beta|^2\rho_t - |\alpha_s|^2\rho_t - |\alpha_r|^2\rho_t$$
$$+ k_1^2\kappa_a(-a^*a\rho_t - \rho_t a^*a) + k_2^2\kappa_b(-b^*b\rho_t - \rho_t b^*b) + k_2^2\kappa_c(-c^*c\rho_t - \rho_t c^*c)$$
$$+ k_1^2\kappa_a a\rho_t a^* + k_1\sqrt{\kappa_a}\,(\beta^* a\rho_t + \rho_t\beta a^*) + |\beta|^2\rho_t + k_1^2\kappa_a a\rho_t a^*$$
$$+ k_2^2 2\kappa_b b\rho_t b^* + k_2\sqrt{2\kappa_b}\,(\alpha_s^* b\rho_t + \rho_t\alpha_s b^*) + |\alpha_s|^2\rho_t + k_2^2 2\kappa_c c\rho_t c^* + k_2\sqrt{2\kappa_c}\,(\alpha_r^* c\rho_t + \rho_t\alpha_r c^*) + |\alpha_r|^2\rho_t,$$

$$\dot{\rho}_t = -i[i\{k_1^2 g_a(a^*\sigma_{ge} - a\sigma_{ge}^*) + k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) + k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) + k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)\}, \rho_t]$$
$$+ \left(-\beta k_1\sqrt{\kappa_a}\,a^* - \alpha_s k_2\sqrt{2\kappa_b}\,b^* - \alpha_r k_3\sqrt{2\kappa_c}\,c^*\right)\rho_t + \rho_t\left(-\beta^* k_1\sqrt{\kappa_a}\,a - \alpha_s^* k_2\sqrt{2\kappa_b}\,b - \alpha_r^* k_3\sqrt{2\kappa_c}\,c\right)$$
$$+ k_1^2\kappa_a(2a\rho_t a^* - a^*a\rho_t - \rho_t a^*a) + k_2^2\kappa_b(2b\rho_t b^* - b^*b\rho_t - \rho_t b^*b) + k_2^2\kappa_c(2c\rho_t c^* - c^*c\rho_t - \rho_t c^*c)$$
$$+ k_1\sqrt{\kappa_a}\,(\beta^* a\rho_t + \rho_t\beta a^*) + k_2\sqrt{2\kappa_b}\,(\alpha_s^* b\rho_t + \rho_t\alpha_s b^*) + k_2\sqrt{2\kappa_c}\,(\alpha_r^* c\rho_t + \rho_t\alpha_r c^*),$$

$$\dot{\rho}_t = -i[i\{k_1^2 g_a(a^*\sigma_{ge} - a\sigma_{ge}^*) + k_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) + k_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) + k_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)\}, \rho_t]$$
$$- i\left[i\left\{k_1\sqrt{\kappa_a}\,(\beta^* a - \beta a^*) + k_2\sqrt{2\kappa_b}\,(\alpha_s^* b - \alpha_s b^*) + k_2\sqrt{2\kappa_c}\,(\alpha_r^* c - \alpha_r c^*)\right\}, \rho_t\right]$$
$$+ k_1^2\kappa_a\{2a\rho_t a^* - a^*a\rho_t - \rho_t a^*a\} + k_2^2\kappa_b\{2b\rho_t b^* - b^*b\rho_t - \rho_t b^*b\} + k_2^2\kappa_c\{2c\rho_t c^* - c^*c\rho_t - \rho_t c^*c\}.$$

Finally we add atomic spontaneous emission terms of all four kinds:

$$\dot{\rho}_t = -i[H, \rho_t] + k_1^2\kappa_a\{2a\rho_t a^* - a^*a\rho_t - \rho_t a^*a\} + k_2^2\kappa_b\{2b\rho_t b^* - b^*b\rho_t - \rho_t b^*b\} + k_2^2\kappa_c\{2c\rho_t c^* - c^*c\rho_t - \rho_t c^*c\}$$
$$+ \gamma_{ge}\{2\sigma_{ge}\rho_t\sigma_{ge}^* - \sigma_{ge}^*\sigma_{ge}\rho_t - \rho_t\sigma_{ge}^*\sigma_{ge}\} + \gamma_{hr}\{2\sigma_{hr}\rho_t\sigma_{hr}^* - \sigma_{hr}^*\sigma_{hr}\rho_t - \rho_t\sigma_{hr}^*\sigma_{hr}\}$$
$$+ \gamma_{gr}\{2\sigma_{gr}\rho_t\sigma_{gr}^* - \sigma_{gr}^*\sigma_{gr}\rho_t - \rho_t\sigma_{gr}^*\sigma_{gr}\} + \gamma_{he}\{2\sigma_{he}\rho_t\sigma_{he}^* - \sigma_{he}^*\sigma_{he}\rho_t - \rho_t\sigma_{he}^*\sigma_{he}\},$$
$$H = ik_1^2 g_a(a^*\sigma_{ge} - a\sigma_{ge}^*) + ik_2 g_b(b^*\sigma_{gr} - b\sigma_{gr}^*) + ik_2 g_b(b^*\sigma_{he} - b\sigma_{he}^*) + ik_2 g_c(c^*\sigma_{hr} - c\sigma_{hr}^*)$$
$$+ ik_1\sqrt{\kappa_a}\,(\beta^* a - \beta a^*) + ik_2\sqrt{2\kappa_b}\,(\alpha_s^* b - \alpha_s b^*) + ik_2\sqrt{2\kappa_c}\,(\alpha_r^* c - \alpha_r c^*).$$

From the adiabatic eliminations studied above, we know that we would like to have $k_1,\ k_2,\ g_b^2/\kappa_b$ and g_c^2/κ_c all large. We presumably also need $k_1 \gg k_2$ in order to justify the order in which we performed the adiabatic eliminations. We thus propose the following master equation for simulation:

$$\dot{\rho}_t = -i[H, \rho_t] + k_1^2\{2a\rho_t a^* - a^*a\rho_t - \rho_t a^*a\} + k_2^2\{2b\rho_t b^* - b^*b\rho_t - \rho_t b^*b\} + k_2^2\{2c\rho_t c^* - c^*c\rho_t - \rho_t c^*c\}$$
$$+ \{2\sigma_{ge}\rho_t\sigma_{ge}^* - \sigma_{ge}^*\sigma_{ge}\rho_t - \rho_t\sigma_{ge}^*\sigma_{ge}\} + \{2\sigma_{hr}\rho_t\sigma_{hr}^* - \sigma_{hr}^*\sigma_{hr}\rho_t - \rho_t\sigma_{hr}^*\sigma_{hr}\}$$
$$+ \{2\sigma_{gr}\rho_t\sigma_{gr}^* - \sigma_{gr}^*\sigma_{gr}\rho_t - \rho_t\sigma_{gr}^*\sigma_{gr}\} + \{2\sigma_{he}\rho_t\sigma_{he}^* - \sigma_{he}^*\sigma_{he}\rho_t - \rho_t\sigma_{he}^*\sigma_{he}\},$$
$$H = ik_1^2(a^*\sigma_{ge} - a\sigma_{ge}^*) + ik_2 g(b^*\sigma_{gr} - b\sigma_{gr}^*) + ik_2 g(b^*\sigma_{he} - b\sigma_{he}^*) + ikg(c^*\sigma_{hr} - c\sigma_{hr}^*)$$
$$+ ik_1(\beta^* a - \beta a^*) + ik_2\sqrt{2}\,(\alpha_s^* b - \alpha_s b^*) + ik_2\sqrt{2}\,(\alpha_r^* c - \alpha_r c^*),$$

to be studied in the regime $k_1 \gg k_2 \gg 1$. It may also be of interest to vary the ratio g, which could be assigned a default value of 1.

Computing finally the output modes, we use the general expression (arXiv:0707.0048v3)

$$(d\mathbf{A}_t)_{out} = \mathbf{S}\,d\mathbf{A}_t + \mathbf{L}dt,$$

where the boldface symbols represent matrices or vectors. Since we are particularly interested in modes 1 and 2 and we have $\mathbf{S} = \mathbf{I}$,

$$L_1 = k_1\sqrt{\kappa_a}\, a^* + \beta^*, \quad M_1 = -k_1\sqrt{\kappa_a}\, a - \beta,$$
$$L_2 = k_1\sqrt{\kappa_a}\, a^*, \quad M_2 = -k_1\sqrt{\kappa_a}\, a,$$

we find

$$(dA_t^1)_{out} = (k_1\sqrt{\kappa_a}\, a^* + \beta^*)dt + dA_t^1,$$
$$(dA_t^2)_{out} = k_1\sqrt{\kappa_a}\, a^* dt + dA_t^2.$$

With the idea that the average intracavity photon number in the empty (no atom) cavity would be

$$\langle a^* a\rangle \to \frac{\left|k_1\sqrt{\kappa_a}\,\beta\right|^2}{k_1^4\kappa_a^2} = \frac{|\beta|^2}{k_1^2\kappa_a},$$

we would expect the output fields out each mirror to have magnitude $k_1\sqrt{\kappa_a}\sqrt{\langle a^* a\rangle} \to \beta$, so this should all work provided the phases do. Explicitly, with the parameter definitions as proposed above we should check

$$\langle (dA_t^1)_{out}\rangle = (k_1\langle a^*\rangle + \beta^*)dt, \quad \langle (dA_t^2)_{out}\rangle = k_1\langle a^*\rangle dt,$$

as α_s and α_r are alternatively activated. We would expect that when α_s is on, the cavity should go transparent and we would have $\langle (dA_t^1)_{out}\rangle = 0$, $\langle (dA_t^2)_{out}\rangle = -\beta^* dt$, while if α_r is on the cavity should go reflective and $\langle (dA_t^1)_{out}\rangle = \beta^* dt$, $\langle (dA_t^2)_{out}\rangle = 0$. Comparing this with the adiabatically eliminated model with

$$L = \begin{pmatrix} \beta^*\Pi_g \\ -\beta^*\Pi_h \\ \alpha_s^*\Pi_h - \alpha_r^*\sigma_{hg} \\ -\alpha_s^*\sigma_{gh} + \alpha_r^*\Pi_g \end{pmatrix}, \quad N = \begin{pmatrix} \Pi_g & -\Pi_h & 0 & 0 \\ -\Pi_h & \Pi_g & 0 & 0 \\ 0 & 0 & \Pi_h & -\sigma_{gh} \\ 0 & 0 & -\sigma_{hg} & \Pi_g \end{pmatrix},$$

we would have

$$(d\mathbf{A}_t)_{out} = \mathbf{S}\, d\mathbf{A}_t + \mathbf{L}dt,$$
$$(dA_t^1)_{out} = \Pi_g dA_t^1 - \Pi_h dA_t^2 + \beta^*\Pi_g\, dt,$$
$$(dA_t^2)_{out} = -\Pi_h dA_t^1 + \Pi_g dA_t^2 - \beta^*\Pi_h\, dt.$$

If $\alpha_s = \alpha_r = 0$ and $\rho = \Pi_h$ (cavity transparency), we would have

$$\langle (dA_t^1)_{out}\rangle = 0, \quad \langle (dA_t^2)_{out}\rangle = -\beta^* dt,$$

while if $\rho = \Pi_g$ (cavity reflective) we would have

$$\langle (dA_t^1)_{out}\rangle = \beta^* dt, \quad \langle (dA_t^2)_{out}\rangle = 0.$$

So that seems to agree fully.