Quantum optical input-output models; quantum optical circuit theory

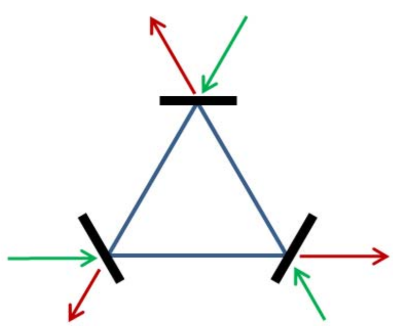
Today we will introduce a formalism for interconnection of quantum optical input-output models of individual components that can be used to derive master equations for a broad class of photonic circuits. The method works for circuits with feedback, but one must take care to avoid feedback topologies that create ‘trapped’ modes, which cannot be handled straightforwardly. The technical approach is based on a somewhat esoteric formulation of non-relativistic quantum field theory in the Markov limit, using quantum stochastic differential equations (QSDEs). It is not necessary to know anything about QSDEs if one simply wants to utilize the methodology to derive circuit master equations from the interconnection of given component models, but next time we will have a quick look ‘under the hood’ at the details of QSDE input-output models in order to establish a very convenient abstraction for nanophotonic circuits that we call the *small volume limit*.

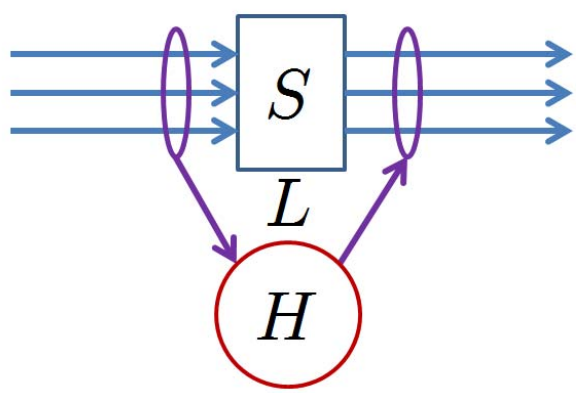
Technical details on the approach can be found in original papers:

- J. Gough and M. R. James, “The Series Product and Its Application to Quantum Feedforward and Feedback Networks,” IEEE Transactions on Automatic Control **54**, 2530 (2009).
- J. Gough and M. R. James, “Quantum Feedback Networks: Hamiltonian Formulation,” Communications in Mathematical Physics **287**, 1109 (2009).

The above papers by Gough and James expand upon some earlier results:

- C. W. Gardiner, “Driving a quantum system with the output of another driven quantum system,” Phys. Rev. Lett. **70**, 2269 (1993).
- H. J. Carmichael, “Quantum trajectory theory for cascaded open quantum systems,” Phys. Rev. Lett. **70**, 2273 (1993).





Each input-output component in a photonic circuit is described by a triple (S, L, H) where S is the scattering matrix of the component, L is the coupling vector of the component, and H is the Hamiltonian of the component’s internal degrees of freedom. S is required to be a unitary matrix, and its matrix elements can in general be operators on the Hilbert space of the component’s internal degrees of freedom (although they are usually just complex numbers). The elements of L can also be operators, in which case they describe the way that external fields couple to the component’s internal degrees of freedom. H plays the usual role of determining the (autonomous) time evolution of the component’s internal degrees of freedom. The dimension of L is equal to the number of input-output ports n that the component has (every port must be both an input and an output); the dimension of S is $n \times n$; the Hamiltonian H is scalar.

Using two composition rules called the series product, denoted $\triangleleft$, and the concatenation product, denoted $\boxplus$, it is possible to combine the individual (S, L, H) models for a number of interconnected components into an overall (S, L, H) model for the whole circuit. Before explaining how the series and concatenation products work, however, let us introduce a few simple component models that we can use for interconnection examples.

A ***beamsplitter*** is a two-input/output component described by a scattering matrix only - it has no internal dynamic degrees of freedom. The (S,L,H) triple is thus simply

$$(S,L,H) = \left(\begin{bmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right),$$

where since the scattering matrix must be unitary,

$$\begin{bmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11}^* & t_{21}^* \\ t_{12}^* & r_{22}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A common example is the 50/50 beamsplitter,

$$\begin{bmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that from a theoretical standpoint the phase convention we have chosen is arbitrary (for example, the transpose of this matrix would equally well represent a 50/50 beamsplitter), but in modeling an actual experimental setup the phases of the matrix elements will be determined by physical properties of the actual beamsplitting device.

An ideal laser input to a photonic circuit can be represented by a ***coherent displacement***, which has the (S,L,H) model

$$(S,L,H) = (1,\alpha,0),$$

where α is the complex amplitude of the displacement (laser input). The units in these models is such that $|\alpha|^2$ should be a flux (photons per second).

A propagation phase or ***phase shift*** can be represented by the (S,L,H) model

$$(S,L,H) = (e^{i\varphi},0,0).$$

Note that this is really a one dimensional scattering matrix. Since the scattering matrix is required to be unitary, it is not valid to consider a component such as $(z,0,0)$ with $|z| \neq 1$. If we want to represent an attenuator of some kind we actually have to pass the field of interest through a beamsplitter, with the second input left open, as this is required at the underlying QSDE level in order to preserve fundamental commutation relations.

An emtpy Fabry-Perot cavity can be represented by an ***optical resonator*** with two input-output ports (two mirrors),

$$(S,L,H) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_1}\,a \\ \sqrt{\kappa_2}\,a \end{bmatrix}, \Delta' a^\dagger a \right).$$

Here Δ accounts for a detuning between the resonance frequency of the cavity eigenmode that is being considered and the rotating frame of the circuit model, κ_1 and κ_2 are the partial decay rates of the two mirrors, and a is the annihilation operator for the cavity eigenmode. If we want to add a two-level atom we can write

$$(S,L,H) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_1}\,a \\ \sqrt{\kappa_2}\,a \\ \sqrt{\gamma}\,\sigma \end{bmatrix}, \Delta' a^\dagger a + \theta |e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) \right),$$

where the extra symbols have their usual meanings from our discussion of cavity QED (although according to the convention for these circuit models, $\kappa_{1,2}$ and γ are energy decay rates rather than field decay rates, thus $\gamma \equiv 2\gamma_\perp$ *et cetera*). Here we have added an extra input-ouput port to represent, in essence, the vacuum electromagnetic fields that interact with the atom and induce spontaneous emission. Note that it would not be realistic to connect the output of this spontaneous emission port to another component because in practice this would require collecting all 4π solid angle of spontaneous emission; on the other hand, we can represent the effect of a laser directly driving the atom (through the ‘side’ of the cavity) by connecting a coherent displacement to the input of this port.

Hopefully it is obvious how to generalize to ring cavities with three or more mirrors, and to cavities with other internal dynamics. For example, a three-mirror ring cavity with a Kerr nonlinear medium is represented by

$$(S,L,H) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_1}\, a \\ \sqrt{\kappa_2}\, a \\ \sqrt{\kappa_3}\, a \end{bmatrix}, \Delta a^\dagger a + \chi a^\dagger a^\dagger a a \right).$$

Note that in some cases we may have a ring cavity in mind in terms of the geometry, yet wish to treat some of the mirrors as ‘high reflectors’ that do not have an associated input-output field. In these cases we can set $\sqrt{\kappa_i} \to 0$ for each such port and in fact the associated row(s) of the scattering matrix and coupling vector can simply be deleted.

To provide at least one example of a more complex device whose scattering matrix contains operator matrix elements, we make note of the ***relay*** model

$$(S,L,H) = \left(\begin{bmatrix} \Pi_g & -\Pi_h & 0 & 0 \\ -\Pi_h & \Pi_g & 0 & 0 \\ 0 & 0 & \Pi_h & -\sigma_{gh} \\ 0 & 0 & -\sigma_{hg} & \Pi_g \end{bmatrix}, 0, 0 \right),$$

where the Hilbert space of the internal degrees of freedom is $\mathrm{span}\{|g\rangle, |h\rangle\}$ and the operators are $\Pi_g = |g\rangle\langle g|$, $\Pi_h = |h\rangle\langle h|$, $\sigma_{hg} = |h\rangle\langle g|$, $\sigma_{gh} = |g\rangle\langle h|$. The first two input-output ports correspond to fields whose routings are switched by the relay, while the third and fourth ports correspond to the SET and RESET inputs of the relay. We’ll see later how to derive such an (S,L,H) as the limit model for a cavity QED system in the small volume limit.

At this point it is useful to note that we can easily write the master equation for any (S,L,H) model (note that we are using $\hbar \to 1$ here):

$$\dot{\rho} = -i[H,\rho] + \sum_j \left(L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j^\dagger L_j \right),$$

where L_j is the j^{th} component of the coupling vector. Note that the scattering matrix does not enter. It is interesting to note that such master equations can sometimes be written in a form that shuffles terms between the Hamiltonian and Lindblad parts – this will help us to match some models we derive using photonic circuit theory with conventional forms from quantum optics, and can also help in making circuit master equations more amenable to numerical integration.

It is clear from inspection that the (S,L,H) cavity QED model reproduces the usual cavity QED master equation using this approach, although without the cavity driving term:

$$(S,L,H) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_1}\, a \\ \sqrt{\kappa_2}\, a \\ \sqrt{\gamma}\, \sigma \end{bmatrix}, \Delta' a^\dagger a + \theta |e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger \sigma) \right),$$

$$\dot{\rho} = -i[H,\rho] + (\kappa_1 + \kappa_2)\left(a\rho a^\dagger - \frac{1}{2} a^\dagger a \rho - \frac{1}{2} \rho a^\dagger a \right) + \gamma \left(\sigma \rho \sigma^\dagger - \frac{1}{2} \sigma^\dagger \sigma \rho - \frac{1}{2} \rho \sigma^\dagger \sigma \right),$$

where we see explicitly the correspondences $\kappa_1 + \kappa_2 \leftrightarrow 2\kappa$ and $\gamma \leftrightarrow 2\gamma_\perp$. But what if we want to add a cavity driving term?

Here we can illustrate the use of the series and concatenation products. The very simple ‘circuit’ model we want is

$$D = C \lhd ((1,\alpha,0) \boxplus (1,0,0) \boxplus (1,0,0)),$$

where C stands for the undriven cavity QED (S,L,H) model written above and D will be the (S,L,H) model for the cavity QED model with a coherent driving field (carrying photon flux $|\alpha|^2$) incident on mirror #1.

We have a general rule for the series product of two systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$:

$$G_2 \triangleleft G_1 = (S_2S_1, S_2L_1 + L_2, H_1 + H_2 + \mathrm{Im}\{L_2^\dagger S_2 L_1\}),$$

and likewise for the concatenation product

$$G_2 \boxplus G_1 = \left(\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H_1 + H_2 \right),$$

where the components of G_1 and G_2 need not commute. Hence we can compute

$$\begin{aligned} D &= \left(I_3, \begin{bmatrix} \sqrt{\kappa_1}\, a \\ \sqrt{\kappa_2}\, a \\ \sqrt{\gamma}\, \sigma \end{bmatrix}, \Delta' a^\dagger a + \theta |e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger \sigma) \right) \triangleleft \left(I_3, \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, 0 \right) \\ &= \left(I_3, \begin{bmatrix} \sqrt{\kappa_1}\, a + \alpha \\ \sqrt{\kappa_2}\, a \\ \sqrt{\gamma}\, \sigma \end{bmatrix}, H_D \right), \end{aligned}$$

where I_3 is the 3×3 identity matrix and

$$\begin{aligned} H_D &= \Delta a^\dagger a + \theta |e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) + \mathrm{Im}\left\{ \begin{bmatrix} \sqrt{\kappa_1}\, a^\dagger & \sqrt{\kappa_2}\, a^\dagger & \sqrt{\gamma}\, \sigma^\dagger \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \right\} \\ &= \Delta' a^\dagger a + \theta |e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) + \mathrm{Im}\{\alpha \sqrt{\kappa_1}\, a^\dagger\} \\ &= \Delta' a^\dagger a + \theta |e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) + \frac{1}{2i}\sqrt{\kappa_1}\,(\alpha a^\dagger - \alpha^* a). \end{aligned}$$

We see that the concatenation product is here used to ‘stack’ three single-field inputs (two of them vacuum) in order to create a G_1 whose dimensions are compatible with those of the cavity QED system G_2. The series product then actually implements driving G_1 ‘into’ G_2 (note that the composition expressions read from right to left).

In order to arrive at a master equation for the driven model, we start by computing the Lindblad terms. Obviously the terms generated by $L_2 = \sqrt{\kappa_2}\, a$ and $L_3 = \sqrt{\gamma}\, \sigma$ won’t change, so we just need to compute

$$\begin{aligned} L_1 \rho L_1^\dagger &= (\sqrt{\kappa_1}\, a + \alpha)\rho(\sqrt{\kappa_1}\, a^\dagger + \alpha^*) \\ &= \kappa_1 a\rho a^\dagger + \sqrt{\kappa_1}\, \alpha^* a\rho + \sqrt{\kappa_1}\, \rho\alpha a^\dagger + |\alpha|^2, \\ L_1^\dagger L_1 &= (\sqrt{\kappa_1}\, a^\dagger + \alpha^*)(\sqrt{\kappa_1}\, a + \alpha) \\ &= \kappa_1 a^\dagger a + \sqrt{\kappa_1}\, \alpha a^\dagger + \sqrt{\kappa_1}\, \alpha^* a + |\alpha|^2, \end{aligned}$$

hence the full Lindblad term for L_1 is

$$\begin{aligned} \{\dot\rho\}_{L_1} &= \kappa_1 a\rho a^\dagger + \sqrt{\kappa_1}\, \alpha^* a\rho + \sqrt{\kappa_1}\, \rho\alpha a^\dagger + |\alpha|^2 \\ &\quad - \frac{1}{2}(\kappa_1 a^\dagger a\rho + \sqrt{\kappa_1}\, \alpha a^\dagger \rho + \sqrt{\kappa_1}\, \alpha^* a\rho + |\alpha|^2\rho) \\ &\quad - \frac{1}{2}(\kappa_1 \rho a^\dagger a + \sqrt{\kappa_1}\, \rho\alpha a^\dagger + \sqrt{\kappa_1}\, \rho\alpha^* a + |\alpha|^2\rho) \\ &= \kappa_1 \left(a\rho a^\dagger - \frac{1}{2} a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a \right) \\ &\quad + \sqrt{\kappa_1}\, \alpha^* a\rho + \sqrt{\kappa_1}\, \rho\alpha a^\dagger - \frac{1}{2}\sqrt{\kappa_1}\, \alpha a^\dagger \rho - \frac{1}{2}\sqrt{\kappa_1}\, \alpha^* a\rho - \frac{1}{2}\sqrt{\kappa_1}\, \rho\alpha a^\dagger - \frac{1}{2}\sqrt{\kappa_1}\, \rho\alpha^* a \\ &= \kappa_1 \left(a\rho a^\dagger - \frac{1}{2} a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a \right) + \frac{1}{2}\sqrt{\kappa_1}\,(\alpha^* a\rho - \alpha a^\dagger \rho + \rho\alpha a^\dagger - \rho\alpha^* a) \\ &= \kappa_1 \left(a\rho a^\dagger - \frac{1}{2} a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a \right) + \frac{1}{2}\sqrt{\kappa_1}\,[(\alpha^* a - \alpha a^\dagger), \rho]. \end{aligned}$$

Thus we see that there is a term we could group into the Hamiltonian, leaving us finally with

$$
\begin{aligned}
\dot{\rho} &= -i[H_D', \rho] + (\kappa_1 + \kappa_2)\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right) + \gamma\left(\sigma\rho\sigma^\dagger - \frac{1}{2}\sigma^\dagger\sigma\rho - \frac{1}{2}\rho\sigma^\dagger\sigma\right), \\
H_D' &= \Delta a^\dagger a + \theta|e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) + \frac{1}{2i}\sqrt{\kappa_1}(\alpha a^\dagger - \alpha^* a) + \frac{1}{(-i)}\frac{1}{2}\sqrt{\kappa_1}(\alpha^* a - \alpha a^\dagger) \\
&= \Delta a^\dagger a + \theta|e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) - \frac{i}{2}\sqrt{\kappa_1}(\alpha a^\dagger - \alpha^* a) + \frac{i}{2}\sqrt{\kappa_1}(\alpha^* a - \alpha a^\dagger) \\
&= \Delta a^\dagger a + \theta|e\rangle\langle e| + g(a\sigma^\dagger + a^\dagger\sigma) - i\sqrt{\kappa_1}(\alpha a^\dagger - \alpha^* a).
\end{aligned}
$$

Hence we finally see that in comparison to the usual form of the driven cavity QED master equation, with cavity driving term

$$H_{dr} = iE(a^\dagger - a),$$

we have derived the correspondence

$$E = -\sqrt{\kappa_1}\,\alpha.$$

If we consider the master equation with $\Delta = \theta = g = 0$ and initial state $|0, g\rangle$ (reducing to a simple resonantly-driven empty cavity)

$$
\begin{aligned}
\frac{d}{dt}\langle a\rangle &= \mathrm{Tr}[a\dot{\rho}] \\
&\to \sqrt{\kappa_1}\,\mathrm{Tr}[a[\alpha^* a - \alpha a^\dagger, \rho]] + (\kappa_1 + \kappa_2)\mathrm{Tr}\left[a\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right)\right] \\
&= \sqrt{\kappa_1}\,\mathrm{Tr}[\alpha^* a a\rho - \alpha a a^\dagger\rho - \alpha^* a\rho a + \alpha a\rho a^\dagger] + (\kappa_1 + \kappa_2)\mathrm{Tr}\left[aa\rho a^\dagger - \frac{1}{2}aa^\dagger a\rho - \frac{1}{2}a\rho a^\dagger a\right] \\
&= \sqrt{\kappa_1}\,\mathrm{Tr}[\alpha^* a a\rho - \alpha a a^\dagger\rho - \alpha^* a a\rho + \alpha a^\dagger a\rho] + \frac{\kappa_1 + \kappa_2}{2}\mathrm{Tr}[a^\dagger a a\rho - a a^\dagger a\rho] \\
&= -\alpha\sqrt{\kappa_1} - \frac{\kappa_1 + \kappa_2}{2}\langle a\rangle,
\end{aligned}
$$

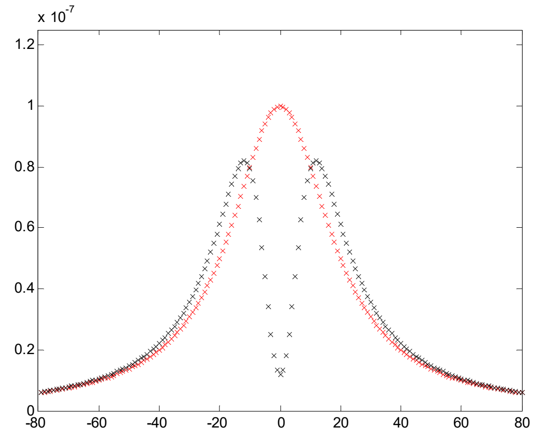
and we therefore find that in steady-state,

$$
\begin{aligned}
\frac{d}{dt}\langle a\rangle &= 0, \\
\langle a\rangle_{ss} &= -\frac{2\alpha\sqrt{\kappa_1}}{\kappa_1 + \kappa_2}.
\end{aligned}
$$

Note that the sign in this expression is crucial. From the underlying QSDE formalism it is possible to derive that for a scenario such as this one with an empty cavity, in which all the optical states should remain simple coherent states, the states of the output channels should be given by the expectation value of the corresponding components of the coupling vector L. Hence in particular, the output state of input-output channels 1 and 2 should be given in steady-state by

$$
\begin{aligned}
\langle L_1\rangle_{ss} &= \langle\sqrt{\kappa_1}\,a + \alpha\rangle_{ss} = -\frac{2\alpha\kappa_1}{\kappa_1 + \kappa_2} + \alpha, \\
\langle L_2\rangle_{ss} &= \langle\sqrt{\kappa_2}\,a\rangle_{ss} = -\frac{2\alpha\sqrt{\kappa_1\kappa_2}}{\kappa_1 + \kappa_2}.
\end{aligned}
$$

We find that if we set $\kappa_1 = \kappa_2$, we recover the expected result for a symmetric two-sided cavity that the reflected power drops to zero while the transmission becomes perfect. Even though the output fields will generally be more complicated that coherent states in the full cavity QED setup with strong coupling, the above expressions remain reasonable indications of the optical power flow in many scenarios where the output states remain close to coherent. Hence for so-called ‘bad cavity’ regime of cavity QED, a numerical computation based on the master equation with $g = 10$ and $\kappa = 20$ produces the following cavity transmission spectrum:



In this plot the horizontal axis is detuning $\Delta' = \theta$ and the vertical axis is actually $\langle a^\dagger a\rangle$ for the intracavity field (red is a $g = 0$ reference, black is $g = 10$). On resonance we see that the intracavity field is very small, and from the above input-output relations we can infer that the incident power is largely reflected (as opposed to dissipated by the atom via spontaneous emission, which would be another reasonable guess). In fact as this spectrum is calculated for a very weak incident probe field, the reflected and transmitted fields should be quite close to coherent states.

Although this has been a very simple first example of using the series and concatenation products, we have already taken a major step forward from our previous discussion involving the master equation only as we now have a rigorous basis for analyzing the internal, reflected and transmitted fields of a cavity QED 'device' based on a specified incident field amplitude. In particular, we know exactly how to relate the parameter E that appeared in our previous master equation to the external amplitude α, and we now how to characterize the partitioning of the total cavity output between reflected and transmitted channels.

To further illustrate the use of series and concatenation products in a more complex scenario, we turn next to an analysis of coherent-feedback suppression of spontaneous switching in ultra-low power dispersive bistability.

We first use the series product to derive an open-loop model for the three-port 'plant' cavity with a coherent driving field. The plant cavity itself is described by an autonomous dynamical model

$$G_b = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_{b2}}\, b \\ \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} \right) = G_{b1} \boxplus G_{b2} \boxplus G_{b3},$$

$$G_{b1} = (1, \sqrt{\kappa_{b1}}\, b, H_{bu}), \quad G_{b2} = (1, \sqrt{\kappa_{b2}}\, b, 0), \quad G_{b3} = (1, \sqrt{\kappa_{b3}}\, b, 0),$$

$$H_{bu} = \chi_b b^\dagger b^\dagger b b + \Delta_b b^\dagger b.$$

In order to include a coherent input field β we use the series product as before,

$$\begin{aligned} N &= G_{b1} \boxplus G_{b2} \boxplus (G_{b3} \lhd (1,\beta,0)) \\ &= G_{b1} \boxplus G_{b2} \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, b^\dagger \beta\}) \\ &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_2}\, b \\ \beta + \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*) \right). \end{aligned}$$

The corresponding open-loop master equation is

$$\dot{\rho} = -i[H,\rho] + \sum_j \left\{ L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j^\dagger L_j \right\}$$

$$\to -i\left[H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*), \rho \right] + (\kappa_{b1} + \kappa_{b2})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2} \rho b^\dagger b \right\}$$

$$+ (\beta + \sqrt{\kappa_{b3}}\, b)\rho(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger) - \frac{1}{2}(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger)(\beta + \sqrt{\kappa_{b3}}\, b)\rho - \frac{1}{2}\rho(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger)(\beta + \sqrt{\kappa_{b3}}\, b)$$

$$= -i\left[H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*), \rho \right] + (\kappa_{b1} + \kappa_{b2})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2} \rho b^\dagger b \right\}$$

$$+ |\beta|^2 \rho + \beta^* \sqrt{\kappa_{b3}}\, b\rho + \beta\sqrt{\kappa_{b3}}\, \rho b^\dagger + \kappa_{b3} b\rho b^\dagger - \frac{1}{2}|\beta|^2\rho - \frac{1}{2}\beta^*\sqrt{\kappa_{b3}}\, b\rho - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\, b^\dagger\rho - \frac{1}{2}\kappa_{b3} b^\dagger b\rho$$

$$- \frac{1}{2}|\beta|^2\rho - \frac{1}{2}\beta^*\sqrt{\kappa_{b3}}\, \rho b - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\, \rho b^\dagger - \frac{1}{2}\kappa_{b3}\rho b^\dagger b$$

$$= -i\left[H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*), \rho \right]$$

$$+ (\kappa_{b1} + \kappa_{b2} + \kappa_{b3})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b \right\} + \frac{1}{2}\beta^*\sqrt{\kappa_{b3}}\,(b\rho - \rho b) - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\,(b^\dagger\rho - \rho b^\dagger).$$

We note that

$$\frac{1}{2}\beta^*\sqrt{\kappa_{b3}}\,(b\rho - \rho b) - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\,(b^\dagger\rho - \rho b^\dagger) = \frac{\sqrt{\kappa_{b3}}}{2}[(\beta^* b - \beta b^\dagger), \rho] = -i\frac{\sqrt{\kappa_{b3}}}{2i}[(\beta b^\dagger - \beta^* b), \rho],$$

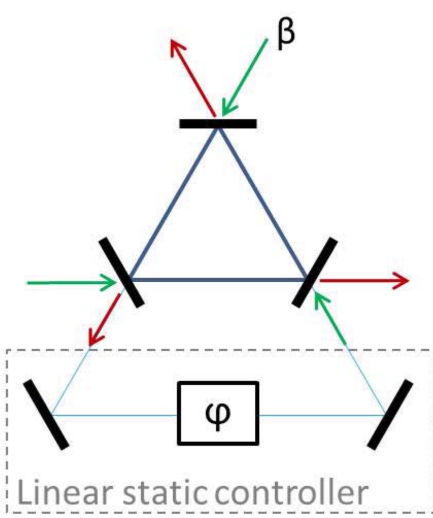
hence we can pull this remaining term into the Hamiltonian and finally write

$$\dot{\rho} = -i[H_{bu} - i\sqrt{\kappa_{b3}}\,(b^\dagger\beta - b\beta^*), \rho] + (\kappa_{b1} + \kappa_{b2} + \kappa_{b3})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b \right\}.$$

We thus see clearly that the total cavity decay rate is simply $\kappa_b \equiv \kappa_{b1} + \kappa_{b2} + \kappa_{b3}$ while the effects of the driving term can be absorbed into the system Hamiltonian. The driven cavity model can thus be written

$$N_d = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_{b2}}\, b \\ \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} - i\sqrt{\kappa_{b3}}\,(b^\dagger\beta - b\beta^*) \right).$$

We next consider the effects of linear static coherent feedback, with a simple phase shift:



We can write

$$
\begin{aligned}
N_{LS} &= G_{b1} \triangleleft ((e^{i\varphi},0,0) \triangleleft G_{b2}) \boxplus (G_{b3} \triangleleft (1,\beta,0)) \\
&= (1,\sqrt{\kappa_{b1}}\,b, H_{bu}) \triangleleft (e^{i\varphi}, e^{i\varphi}\sqrt{\kappa_{b2}}\,b, 0) \boxplus (G_{b3} \triangleleft (1,\beta,0)) \\
&= (e^{i\varphi}, (\sqrt{\kappa_{b1}} + e^{i\varphi}\sqrt{\kappa_{b2}})b, H_{bu} + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\,b^\dagger b) \boxplus (G_{b3} \triangleleft (1,\beta,0)) \\
&= \left(\begin{bmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} (\sqrt{\kappa_{b1}} + e^{i\varphi}\sqrt{\kappa_{b2}})b \\ \sqrt{\kappa_{b3}}\,b \end{bmatrix}, H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*) + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\,b^\dagger b \right),
\end{aligned}
$$

where we re-use what we have derived above regarding the driving term, yielding the closed-loop master equation

$$
\begin{aligned}
\dot{\rho} = &-i[H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*) + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\,b^\dagger b, \rho] \\
&+ \left(\kappa_{b3} + |\sqrt{\kappa_{b1}} + e^{i\varphi}\sqrt{\kappa_{b2}}|^2\right)\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\}.
\end{aligned}
$$

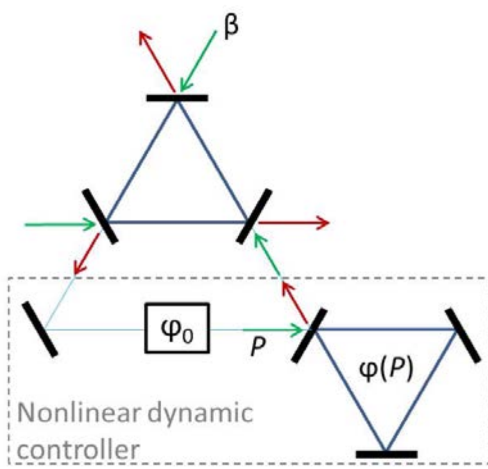
Hence the total cavity decay rate is a function of φ, and there is an additional frequency-pulling term in the Hamiltonian. We note that for $\varphi = 0$ we obtain

$$
\dot{\rho} \to -i[H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*), \rho] + (\kappa_b + 2\sqrt{\kappa_{b1}\kappa_{b2}})\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\},
$$

while for $\varphi = \pi$ we obtain

$$
\dot{\rho} \to -i[H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*), \rho] + (\kappa_b - 2\sqrt{\kappa_{b1}\kappa_{b2}})\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\}.
$$

Hence in these simple cases we have either a pure increase or a pure decrease in the cavity decay rate as the only net effect of the feedback. These can be understood as interferometric constructive/destructive interference of the output fields from the κ_{b1} and κ_{b2} cavity mirrors. We infer that since the external driving term (through mirror κ_{b3}) is unaffected, it should be possible to use φ to tune the average intracavity photon number. In particular if we have a detuned driving field, we should be able to decrease the effective driving strength by decreasing the effective κ_b and vice versa.



For the nonlinear dynamic controller we assume two cavities a (controller) and b (plant) with component models

$$
G_a = (1, \sqrt{\kappa_a}\,a, H_a),
$$

$$
G_b = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\,b \\ \sqrt{\kappa_{b2}}\,b \\ \sqrt{\kappa_{b3}}\,b \end{bmatrix}, H_{bu} \right),
$$

where

$$
\begin{aligned}
H_a &= \chi_a a^\dagger a^\dagger a a + \Delta_a a^\dagger a, \\
H_{bu} &= \chi_b b^\dagger b^\dagger b b + \Delta_b b^\dagger b.
\end{aligned}
$$

We define the partitioning

$$G_b = G_{b1} \boxplus G_{b2} \boxplus G_{b3},$$

where

$$G_{b1} = (1, \sqrt{\kappa_{b1}}\, b, H_{bu}), \quad G_{b2} = (1, \sqrt{\kappa_{b2}}\, b, 0), \quad G_{b3} = (1, \sqrt{\kappa_{b3}}\, b, 0).$$

We compute the feedback network as

$$\begin{aligned}
N_{ND} &= G_{b1} \lhd (G_a \lhd ((e^{i\varphi_0}, 0, 0) \lhd G_{b2})) \boxplus (G_{b3} \lhd (1, \beta, 0)) \\
&= ((1, \sqrt{\kappa_{b1}}\, b, H_{bu}) \lhd ((1, \sqrt{\kappa_a}\, a, H_a) \lhd (e^{i\varphi}, e^{i\varphi}\sqrt{\kappa_{b2}}\, b, 0))) \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, \beta b^\dagger\}) \\
&= ((1, \sqrt{\kappa_{b1}}\, b, H_{bu}) \lhd (e^{i\varphi}, \sqrt{\kappa_a}\, a + e^{i\varphi}\sqrt{\kappa_{b2}}\, b, H_a + \mathrm{Im}\{e^{i\varphi}\sqrt{\kappa_a \kappa_{b2}}\, a^\dagger b\})) \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, \beta b^\dagger\}) \\
&= \big(e^{i\varphi}, \sqrt{\kappa_a}\, a + (e^{i\varphi}\sqrt{\kappa_{b2}} + \sqrt{\kappa_{b1}})b,\ H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b + \mathrm{Im}\{e^{i\varphi}\sqrt{\kappa_a \kappa_{b2}}\, a^\dagger b + \sqrt{\kappa_a \kappa_{b1}}\, a b^\dagger\}\big) \\
&\quad \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, \beta b^\dagger\}) \\
&= \left(S_{ND}, \begin{bmatrix} \sqrt{\kappa_a}\, a + (e^{i\varphi}\sqrt{\kappa_{b2}} + \sqrt{\kappa_{b1}})b \\ \beta + \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b + \mathrm{Im}\{e^{i\varphi}\sqrt{\kappa_a \kappa_{b2}}\, a^\dagger b + \sqrt{\kappa_a \kappa_{b1}}\, a b^\dagger + \sqrt{\kappa_{b3}}\, \beta b^\dagger\} \right).
\end{aligned}$$

We thus have a total Hamiltonian,

$$H = H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b + \frac{\sqrt{\kappa_a \kappa_{b2}}}{2i}(e^{i\varphi} a^\dagger b - e^{-i\varphi} a b^\dagger) + \frac{\sqrt{\kappa_a \kappa_{b1}}}{2i}(a b^\dagger - a^\dagger b) + \frac{\sqrt{\kappa_{b3}}}{2i}(\beta b^\dagger - \beta^* b),$$

and (as we did above) we note that the second Lindblad term leads to terms in the Master Equation,

$$\begin{aligned}
[\dot{\rho}]_{L_2} &= L_2 \rho L_2^\dagger - \frac{1}{2} L_2^\dagger L_2 \rho - \frac{1}{2} \rho L_2^\dagger L_2 \\
&= (\beta + \sqrt{\kappa_{b3}}\, b)\rho(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger) - \frac{1}{2}(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger)(\beta + \sqrt{\kappa_{b3}}\, b)\rho - \frac{1}{2}\rho(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger)(\beta + \sqrt{\kappa_{b3}}\, b) \\
&= |\beta|^2 \rho + \beta \sqrt{\kappa_{b3}}\, \rho b^\dagger + \beta^* \sqrt{\kappa_{b3}}\, b\rho + \kappa_{b3} b \rho b^\dagger - \frac{1}{2}\big(|\beta|^2 + \beta^* \sqrt{\kappa_{b3}}\, b + \beta \sqrt{\kappa_{b3}}\, b^\dagger + \kappa_{b3} b^\dagger b\big)\rho \\
&\quad - \frac{1}{2}\rho\big(|\beta|^2 + \beta^* \sqrt{\kappa_{b3}}\, b + \beta \sqrt{\kappa_{b3}}\, b^\dagger + \kappa_{b3} b^\dagger b\big) \\
&= \kappa_{b3}\left\{b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2}\rho b^\dagger b\right\} + \frac{1}{2}\beta \sqrt{\kappa_{b3}}\, \rho b^\dagger + \frac{1}{2}\beta^* \sqrt{\kappa_{b3}}\, b\rho - \frac{1}{2}\beta \sqrt{\kappa_{b3}}\, b^\dagger \rho - \frac{1}{2}\rho \beta^* \sqrt{\kappa_{b3}}\, b \\
&= \kappa_{b3}\left\{b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2}\rho b^\dagger b\right\} + \frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger)\rho + \frac{\sqrt{\kappa_{b3}}}{2}\rho(\beta b^\dagger - \beta^* b).
\end{aligned}$$

We retain the first term in braces as a modified $L_2 \to \sqrt{\kappa_{b3}}\, b$ and note that

$$\begin{aligned}
\frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger)\rho + \frac{\sqrt{\kappa_{b3}}}{2}\rho(\beta b^\dagger - \beta^* b) &= \left[\frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger), \rho\right] \\
&= -i\left[i\frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger), \rho\right] \\
&= -i\left[\frac{\sqrt{\kappa_{b3}}}{2i}(\beta b^\dagger - \beta^* b), \rho\right].
\end{aligned}$$

We therefore add this to the original Hamiltonian terms to obtain

$$\begin{aligned}
H &\to H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b + \frac{\sqrt{\kappa_a \kappa_{b2}}}{2i}(e^{i\varphi} a^\dagger b - e^{-i\varphi} a b^\dagger) + \frac{\sqrt{\kappa_a \kappa_{b1}}}{2i}(a b^\dagger - a^\dagger b) + i\sqrt{\kappa_{b3}}\,(\beta^* b - \beta b^\dagger), \\
L_1 &= \sqrt{\kappa_a}\, a + (e^{i\varphi}\sqrt{\kappa_{b2}} + \sqrt{\kappa_{b1}})b, \\
L_2 &= \sqrt{\kappa_{b3}}\, b.
\end{aligned}$$