Simulating quantum computers with probabilistic methods

Tokyo Impact Lecture 1: Theory of phase-space representations

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Outline

1. Quantum dynamics
2. Number state evolution
3. Classical phase-space
4. Wigner stochastic equations
5. Non-classical phase-space
6. Examples

Photonics and ultracold atoms - ideal quantum systems

Photons: $\hbar\omega \ll kT$; Atoms: ULTRALOW temperatures to $1nK$

What is different about photons and ultracold atoms?

- Photons have weak interactions with dielectrics
- Retain quantum coherence over long distances
- Atoms can be trapped in a hard vacuum
- Cooling to nanoKelvins or less
- Correlations - mean field theory doesn't work
- Dynamics - time-evolution is very important

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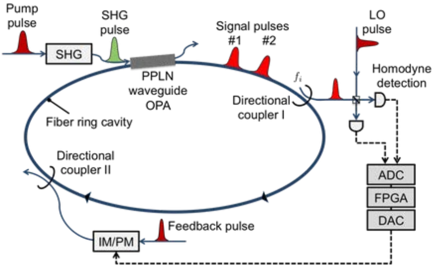
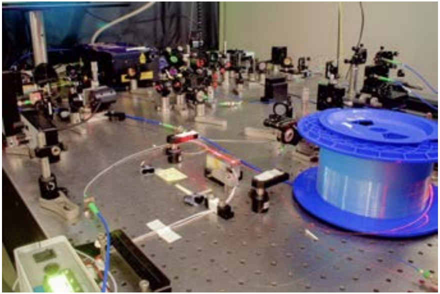
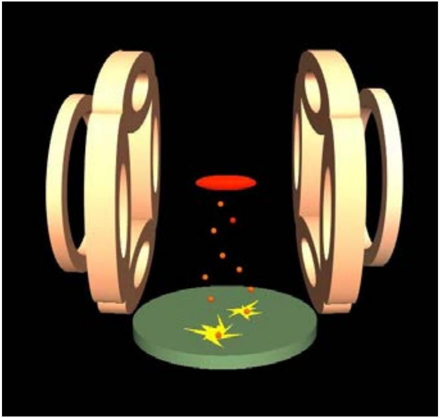
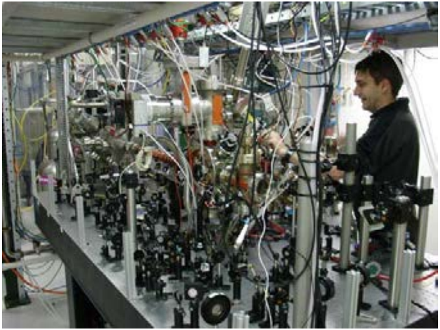
Typical photonic experiment (Tokyo, Stanford)





Typical atomic experiment (Orsay, ANU)





How to calculate dynamics?

Classical solution: - use Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Quantum mechanics replaces classical quantities with operators:

$$[\widehat{q}_i, \widehat{p}_j] = i\hbar\delta_{ij}$$
$$[\widehat{q}_i, \widehat{q}_j] = [\widehat{p}_i, \widehat{p}_j] = 0$$

Then, for any operator $\hat{O}$, in the Heisenberg picture:

$$\frac{\partial \hat{O}}{\partial t} = \frac{1}{i\hbar}\left[\widehat{O}, \widehat{H}\right]$$

What about mixtures of states?

Suppose the quantum system is in a mixture of quantum states $|\psi_m\rangle$ with probability p_m . Then the density matrix $\hat{\rho}$ is defined as:

$$\hat{\rho} = \sum_m p_m |\psi_m\rangle \langle\psi_m|$$

In the Schroedinger picture, we let states evolve in time, not operators!

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{1}{i\hbar}\left[\widehat{H}, \widehat{\rho}\right]$$

Then, for any operator $\hat{O}$, the expectation value of the observable is:

$$\left\langle \hat{O} \right\rangle = Tr\left[\hat{\rho}\,\widehat{O}\right]$$

What is the quantum field Hamiltonian anyway?

Here $\widehat{\Psi}_i$ is a bosonic field of spin/frequency index i:

$$\left[\widehat{\Psi}_i(\mathbf{x}), \widehat{\Psi}_i^\dagger(\mathbf{x}')\right]_\pm = \delta_{ij}\delta^D(\mathbf{x}-\mathbf{x}')$$

In second quantization the quantum Hamiltonian is

$$\begin{aligned}\widehat{H} &= \sum_i \int d^D\mathbf{x} \left\{ \frac{\hbar^2}{2m} \nabla \widehat{\Psi}_i^\dagger(\mathbf{x}) \cdot \nabla \widehat{\Psi}_i(\mathbf{x}) + U_i^{(1)}(\mathbf{x}) \widehat{\Psi}_i^\dagger(\mathbf{x}) \widehat{\Psi}_i(\mathbf{x}) \right\} \\ & \sum_{ijk} \int d^D\mathbf{x} \left\{ U_{ijk}^{(2)}(\mathbf{x}) \widehat{\Psi}_i^\dagger(\mathbf{x}) \widehat{\Psi}_j(\mathbf{x}) \widehat{\Psi}_k(\mathbf{x}) + h.c. \right\} \\ &+ \sum_{ij} \frac{1}{2} \int d^D\mathbf{x} U_{ij}^{(3)}(\mathbf{x}) \widehat{\Psi}_i^\dagger(\mathbf{x}) \widehat{\Psi}_j^\dagger(\mathbf{x}) \widehat{\Psi}_j(\mathbf{x}) \widehat{\Psi}_i(\mathbf{x}) .\end{aligned}$$

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What are the parameters?

This describes a dilute gas of photons or atoms:

- $\langle \widehat{\Psi}_i^\dagger(\mathbf{x}) \widehat{\Psi}_i(\mathbf{x}) \rangle$ is the spin i particle density,
- m is the effective mass, or equivalent dispersion coefficient
- $U_i^{(1)}$ is the trapping potential
- $U_{ijk}^{(2)}$ is the parametric coupling for downconversion
- $U_{ij}^{(3)}$ is $\chi^{(3)}$ for photons, S-wave scattering for atoms
- Here we implicitly assume a momentum cutoff k_c

How do we treat quantum fields?

Any field operator $\hat{\Psi}$ can be expanded in orthogonal modes:

$$\hat{\Psi}(\mathbf{x}) = \sum \hat{a}_m u_m(\mathbf{x})$$

Where: $\int d^3\mathbf{x}\; u_m^*(\mathbf{x})\, u_n(\mathbf{x}) = \delta_{mn}$

Nonvanishing field (anti)-commutators are given by:

$$\left[\hat{\Psi}(\mathbf{x}), \hat{\Psi}^\dagger(\mathbf{x}')\right]_\pm = \delta^3(\mathbf{x} - \mathbf{x}')$$

$(+)$ = anticommutator (FERMION) and

$(-)$ = commutator (BOSON)

Local Mode Operators

Assume that the mode operators are localized on a lattice

Take a discrete Fourier transform to get localized modes
Spin and position indices $= \{s_k, \mathbf{r}_k\}$ with lattice volume ΔV:

$$\hat{a}_i = \sqrt{\Delta V}\widehat{\Psi}_{s_k \mathbf{r}_k}$$

In the case of bosonic (fermionic) fields, the commutators (anticommutators) are defined as:

$$\left\{\hat{a}_i, \hat{a}_j^\dagger\right\}_\pm = \delta_{ij}$$

The Hamiltonian is exact for a large number of sites:

$$\widehat{H}(\hat{\boldsymbol{a}}^\dagger, \hat{\boldsymbol{a}}) \approx \hbar\left[\omega_{ij}\hat{a}_i^\dagger\hat{a}_j + \chi_{ijk}\widehat{a}_i^\dagger\widehat{a}_j\widehat{a}_k + \frac{1}{2}\kappa_{ij}\widehat{n}_i\widehat{n}_j\right].$$

What do the mode operators do?

Bosons $\leftrightarrow$ harmonic oscillators; fermions $\leftrightarrow$ two-level atoms

$$\hat{a}^\dagger |N\rangle = \delta_N |N+1\rangle \quad (FERMION)$$
$$\hat{a}^\dagger |N\rangle = \sqrt{N+1}|N+1\rangle \quad (BOSON)$$
$$\hat{a}|N\rangle = \sqrt{N}|N-1\rangle$$

Hence the single mode number operator is $\hat{N} = \hat{a}^\dagger \hat{a}$:

$$\hat{N}|N\rangle = \hat{a}^\dagger \hat{a}|N\rangle = \hat{a}^\dagger \sqrt{N}|N-1\rangle = N|N\rangle$$

In the FERMION case, you can only have $N = 0, 1$

What quantum states can we have?

Quantum states are generated from the vacuum state

- Number states:

$$|N_1,\dots N_m\rangle = \frac{(\psi_1)^{N_1}\dots(a_m^\dagger)^{N_m}}{\sqrt{N_1!\dots N_m!}}|0\rangle$$

- Properties:

$$\langle \mathbf{M} | \, \mathbf{N}\rangle = \delta_{N_1 M_1}\dots\delta_{N_m M_m}$$

- Fermion case: must have $N_j = 0, 1$

All other states can be generated using linear combinations

Example: single mode coherent state

Single mode coherent state has a well-defined phase

- Boson coherent state:

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - |\alpha|^2/2}|0\rangle = e^{-|\alpha|^2/2}\sum_{N=0}^{\infty}\frac{\alpha^N}{\sqrt{N!}}|N\rangle$$

- Fermion coherent state: requires Grassmann algebra (not treated)

Important properties:

$$|\langle\alpha|\,\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$$
$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$
$$\langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle = |\alpha|^2$$

Simplest method for state evolution

Suppose the quantum system is described by a few modes:

$$|\psi\rangle = \sum \psi_{\mathbf{N}} |N_1, N_2, \ldots N_m\rangle = \sum \psi_{\mathbf{N}} |\mathbf{N}\rangle$$

Then, let $H_{\mathbf{NM}} = \langle \mathbf{N} | \hat{H} | \mathbf{M} \rangle$ and: $\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} \hat{H} |\psi\rangle$

Hence, we have a simple matrix equation:

$$\frac{d}{dt} \psi_{\mathbf{N}} = -\frac{i}{\hbar} \sum_{\mathbf{M}} H_{\mathbf{NM}} \psi_{\mathbf{M}}$$

Problem: quantum theory is exponentially complex!

Quantum many-body problems are very large

- consider N particles distributed among M modes
- take $N \simeq M \simeq 500,000$:
- Number of quantum states: $N_s = 2^{2N} = 2^{1,000,000}$
- **More quantum states than atoms in the universe**
- How big is your computer?
- **Can't diagonalize $2^{1,000,000} \times 2^{1,000,000}$ Hamiltonian!**

What about losses and damping?

Damping can be treated using a master equation

- The density matrix $\hat{\rho}$ evolves as:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}\right] + \sum_j \kappa_j \int d^3\mathbf{r}\mathscr{L}_j\left[\hat{\rho}\right]$$

- Here the Liouville terms describe coupling to the reservoirs:

$$\mathscr{L}_j\left[\hat{\rho}\right] = 2\hat{O}_j\hat{\rho}\hat{O}_j^\dagger - \hat{O}_j^\dagger\hat{O}_j\hat{\rho} - \hat{\rho}\hat{O}_j^\dagger\hat{O}_j$$

- For n-particle collisions: $\hat{O}_i = \left[\widehat{\Psi}_i(\mathbf{r})\right]^n$

Traditional quantum theory methods?

- numerical diagonalisation?
  intractable for $\gtrsim 10$ modes
- operator factorization
  not applicable for strong correlations
- perturbation theory
  diverges at strong couplings, large order
- exact solutions
  not usually applicable to quantum dynamics

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Quantum theory in classical phase-space

Properties of Wigner/Moyal phase-space

- Maps quantum states into **classical phase-space** $\alpha = p + ix$
- **Wigner first published this representation**
- Moyal showed equivalence to quantum mechanics
- Complexity grows only linearly with number of modes!

Problem: Wigner distribution can have negative values

- **Need to truncate equations to get positive probabilities**

Detailed equivalence

Mapping of characteristic functions

$$W(\boldsymbol{\alpha}) = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z} \left\langle e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^{*}\cdot\left(\hat{\mathbf{a}}^{\dagger}-\boldsymbol{\alpha}^{*}\right)} \right\rangle$$

Operator mean values

- $\left\langle \hat{a}_i^{\dagger m} \hat{a}_j^{n} \right\rangle_{SYM} = \int d^{2M}\boldsymbol{\alpha}\, \alpha_i^{*m} \alpha_j^{n} W(\boldsymbol{\alpha}) = \left\langle \alpha_i^{*m} \alpha_j^{n} \right\rangle_W$
- $\langle \hat{a}_j \rangle = \langle \alpha_j \rangle_W$
- $\left\langle \hat{a}_i^{\dagger} \hat{a}_j + \hat{a}_i \hat{a}_j^{\dagger} \right\rangle / 2 = \langle \alpha_i^{*} \alpha_j \rangle_W$

Detailed equivalence

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Dynamical equivalence

Mapping of dynamical equations

$$\frac{\partial W(\boldsymbol{\alpha})}{\partial t} = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z}\, Tr\left[\frac{\partial \hat{\rho}}{\partial t} e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^{*}\cdot\left(\hat{\mathbf{a}}^{\dagger}-\boldsymbol{\alpha}^{*}\right)}\right]$$

Operator mappings

- $\hat{a}_j\hat{\rho} \to \left(\alpha_j + \frac{1}{2}\frac{\partial}{\partial \alpha_j^*}\right) W$
- $\hat{\rho}\hat{a}_j^{\dagger} \to \left(\alpha_j^* + \frac{1}{2}\frac{\partial}{\partial \alpha_j}\right) W$
- $\hat{a}_j^{\dagger}\hat{\rho} \to \left(\alpha_j^* - \frac{1}{2}\frac{\partial}{\partial \alpha_j}\right) W$
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$$\psi(\mathbf{x}) = \sum \alpha_m u_m(\mathbf{x})$$

Where: $\int d^3\mathbf{x}\ u_m^*(\mathbf{x})\, u_n(\mathbf{x}) = \delta_{mn}$

Nonvanishing vacuum expectation values are given by:

$$\left\langle \psi^*(\mathbf{x})\,\psi(\mathbf{x}')\right\rangle = \frac{1}{2}\delta^3(\mathbf{x}-\mathbf{x}')$$

Example: Wigner function for a coherent state

Suppose we have a single mode system in a coherent state

$$\hat{\rho} = |\alpha_0\rangle\langle\alpha_0|$$

Hence:

$$W(\alpha) = \frac{1}{\pi^2}\int d^2z \langle\alpha_0| e^{iz\cdot(\hat{a}-\alpha)+iz\cdot\left(\hat{a}^\dagger-\alpha^*\right)}|\alpha_0\rangle$$

Solution with a little algebra

- $$W(\alpha) = \frac{2}{\pi}e^{-2|\alpha-\alpha_0|^2}$$

This solution gives $\langle\alpha^*\alpha\rangle = 1/2$ for a vacuum state

Example: time-evolution of harmonic oscillator

Consider the harmonic oscillator

$$\hat{H} = \hbar\omega\hat{a}^{\dagger}\hat{a}$$
$$\frac{\partial\hat{\rho}}{\partial t} = -i\omega\left[\hat{a}^{\dagger}\hat{a}\widehat{\rho} - \widehat{\rho}\hat{a}^{\dagger}\hat{a}\right]$$

Operator mappings

- $\hat{a}^{\dagger}\hat{a}\hat{\rho} \rightarrow \left(\alpha^{*} - \frac{1}{2}\frac{\partial}{\partial\alpha}\right)\left(\alpha + \frac{1}{2}\frac{\partial}{\partial\alpha^{*}}\right)W$
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- $$\frac{\partial W}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^{*}}\alpha^{*}\right)W$$

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- $$\frac{\partial W}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^*\right)W$$

Harmonic oscillator solution

General result for harmonic oscillator

$$\frac{\partial W}{\partial t} = i\omega \left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^* \right) W$$

Solution by method of characteristics

- $$\frac{\partial \alpha}{\partial t} = -i\omega\alpha$$
- $$\alpha(t) = \alpha(0)e^{-i\omega t}$$

Fokker-Planck equations

Result of operator mappings:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} + \frac{1}{6} \frac{\partial^3}{\partial \alpha_i \partial \alpha_j^* \partial \alpha_k^*} T_{ijk} + \dots \right\} W$$

Scaling to eliminate higher-order terms

$$x = \alpha / \sqrt{N}$$

$$\frac{\partial W}{\partial t} = \left\{ -\frac{1}{\sqrt{N}} \frac{\partial}{\partial x_i} A_i + \frac{1}{2N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} + O\left(\frac{1}{N^{3/2}} \right) \right\} W$$

Stochastic equation

Result of operator mappings + truncation - valid if $N/M >> 1$:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} \right\} W$$

Equivalent stochastic equation

$$\frac{\partial \alpha_i}{\partial t} = A_i + \zeta_i(t)$$

where:

$$\left\langle \zeta_i(t) \zeta_j^*(t) \right\rangle = D_{ij} \delta \left(t - t' \right)$$

Example: optical fiber/BEC case

Result of operator mappings + truncation - for the GPE:

$$\frac{d\psi_j}{dt} = iK_j\psi_j - iU_{ij}^{(3)}|\psi_i|^2\psi_j - \gamma_j\psi_j + \sqrt{\gamma_j}\zeta_j(\mathbf{x},t)$$

Here the linear unitary evolution of the wave-function, is described by:

$$K_j = \hbar\nabla^2/2m - U_j^{(1)}(\mathbf{r})$$

while $\zeta_i(\mathbf{x},t)$ is a complex, stochastic delta-correlated Gaussian noise with

$$\left\langle \zeta_i(\mathbf{x},t)\zeta_j^*(\mathbf{x}',t')\right\rangle = \delta_{ij}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right).$$

Initial fluctuations: $\langle\Delta\Psi_s(\mathbf{x})\Delta\Psi_u^*(\mathbf{x}')\rangle = \frac{1}{2}\delta_{su}\delta^3(\mathbf{x}-\mathbf{x}')$

Parametric waveguide

Result of operator mappings + truncation

$$\frac{d\psi_0}{dt} = -\gamma_0\psi_0 + \mathscr{E}(x) - \chi\psi_1\psi_2 + \frac{iv_0^2}{2\omega_0}\nabla^2\psi_0 + \sqrt{\gamma_0}\zeta_0$$
$$\frac{d\psi_1}{dt} = -\gamma_1\psi_1 + \chi\psi_0\psi_2^* + \frac{iv_1^2}{2\omega_1}\nabla^2\psi_1 + \sqrt{\gamma_1}\zeta_1$$
$$\frac{d\psi_2}{dt} = -\gamma_2\psi_2 + \chi\psi_0\psi_1^* + \frac{iv_2^2}{2\omega_2}\nabla^2\psi_2 + \sqrt{\gamma_2}\zeta_2$$

$\zeta_i(t)$ is a complex, stochastic delta-correlated Gaussian noise:

$$\left\langle \zeta_i(\mathbf{x},t)\zeta_j^*(\mathbf{x}',t')\right\rangle = \delta_{ij}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right).$$

First-principle simulations: the positive-P representation

What do we do with modes having low occupation numbers?

- Truncated Wigner only works if all modes are heavily occupied
- How about modeling other cases with low occupations:
  - Example: formation of a BEC must start with low occupation!
  - collisions that generate atoms in initially empty modes
  - coupling to thermal modes having low occupation?
- We need a technique without the large N approximation
- The positive-P representation does not truncate terms

+P PHASE-SPACE METHODS

The positive P-representation expands in coherent state projectors

$$\widehat{\rho} = \int P(\boldsymbol{\alpha},\boldsymbol{\beta})\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta})d^{2M}\boldsymbol{\alpha}d^{2M}\boldsymbol{\beta}$$

$$\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\beta}^*|}{\langle\boldsymbol{\beta}^*||\boldsymbol{\alpha}\rangle}$$

Enlarged phase-space allows positive probabilities!

- Maps quantum states into 4M real coordinates: $\boldsymbol{\alpha},\boldsymbol{\beta} = \mathbf{p}+i\mathbf{x},\mathbf{p}'+i\mathbf{x}'$
- Double the size of a classical phase-space
- Exact mappings even for low occupations
- Advantage: Can represent entangled states

+P PHASE-SPACE METHODS

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+P Existence Theorem

For ANY density matrix, a positive P-function always exists

$$P(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{(2\pi)^{2M}} e^{-\left|\boldsymbol{\alpha}-\boldsymbol{\beta}^*\right|^2/4} \left\langle \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2} \middle| \widehat{\rho} \middle| \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2} \right\rangle$$

Enlarged phase-space allows positive probabilities!

- Advantage: Probabilistic sampling is possible
- Problem: Non-uniqueness may allows sampling error to grow in time

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Operator identities

Differentiating the projection operator gives the following identities

$$\begin{aligned}
\widehat{a}_n^{\dagger}\widehat{\rho} &\rightarrow \left[\beta_n - \frac{\partial}{\partial \alpha_n}\right] P \\
\widehat{a}_n\widehat{\rho} &\rightarrow \alpha_n P \\
\widehat{\rho}\widehat{a}_n &\rightarrow \left[\alpha_n - \frac{\partial}{\partial \beta_n}\right] P \\
\widehat{\rho}\widehat{a}_n^{\dagger} &\rightarrow \beta_n P
\end{aligned}$$

Since the projector is an analytic function of both α_n and β_n, we can obtain alternate identities by replacing $\partial/\partial\alpha$ by either $\partial/\partial\alpha_x$ or $\partial/i\partial\alpha_y$. This equivalence allows a positive-definite diffusion to be obtained, with stochastic evolution.

How do we calculate an operator expectation value

- There is a correspondence between the moments of the distribution, and the normally ordered operator products.
- These come from the fact that coherent states are eigenstates of the annihilation operator
- Using $\mathrm{Tr}\left[\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta})\right]=1$:

$$\langle \widehat{a}_m^\dagger \cdots \widehat{a}_n \rangle = \int\int P(\boldsymbol{\alpha},\boldsymbol{\beta})[\beta_m \cdots \alpha_n] d^{2M}\boldsymbol{\alpha}\, d^{2M}\boldsymbol{\beta}\,.$$

Example: time-evolution of harmonic oscillator

Consider the harmonic oscillator

$$\hat{H} = \hbar\omega \hat{a}^{\dagger}\hat{a}$$

$$\frac{\partial \hat{\rho}}{\partial t} = -i\omega\left[\hat{a}^{\dagger}\hat{a}\widehat{\rho} - \widehat{\rho}\hat{a}^{\dagger}\hat{a}\right]$$

Operator mappings

- $\hat{a}^{\dagger}\hat{a}\hat{\rho} \to \left[\beta - \frac{\partial}{\partial\alpha}\right]\alpha P$
- $\hat{\rho}\hat{a}^{\dagger}\hat{a} \to \left[\alpha - \frac{\partial}{\partial\beta}\right]\beta P$
- $$\frac{\partial P}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\beta}\beta\right)P$$

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Harmonic oscillator solution

General result for harmonic oscillator

$$\frac{\partial P}{\partial t} = i\omega \left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \beta}\beta \right) P$$

Solution by method of characteristics

- $$\frac{d\alpha}{dt} = -i\omega\alpha$$
- $$\alpha(t) = \alpha(0)e^{-i\omega t}$$

The linear time evolution is exactly the same as for a Wigner function

For coherent states, initial condition is a delta function, not a Gaussian.

General case

Take a more general Hamiltonian, with nonlinear terms

Then we define

$$\overrightarrow{\alpha} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \equiv (\boldsymbol{\alpha}, \boldsymbol{\alpha}^{+})$$

and find using operator mappings that - provided the distribution is sufficiently bounded at infinity:

$$\frac{\partial}{\partial t} P(t, \overrightarrow{\alpha}) = \left[\partial_i A_i(\overrightarrow{\alpha}) + \frac{1}{2} \partial_i \partial_j D_{ij}(t, \overrightarrow{\alpha}) \right] P(t, \overrightarrow{\alpha}).$$

Comparison of positive-P and Wigner

- There are no other terms in +P - higher order derivatives all vanish
- Nonlinear couplings cause noise, linear damping does not

How do we treat +P fields?

Field operators come in conjugate pairs:

$$\psi(\mathbf{x}) = \sum \alpha_m u_m(\mathbf{x})$$
$$\psi^{+}(\mathbf{x}) = \sum \alpha_m^{+} u_m^{*}(\mathbf{x})$$

Where: $\int d^3\mathbf{x}\ u_m^{*}(\mathbf{x})\, u_n(\mathbf{x}) = \delta_{mn}$
Vacuum expectation values are given by:

$$\left\langle \psi^{+}(\mathbf{x})\, \psi(\mathbf{x}')\right\rangle = 0$$

+P equations for the fiber/BEC case

Exact result of operator mappings - assume U_{ij} is diagonal

$$\frac{d\psi_j}{dt} = iK_j\psi_j - iU_j^{(3)}\psi_j^+\psi_j^2 + \sqrt{-iU_j^{(3)}}\psi_j\zeta_j(t)$$
$$\frac{d\psi_j^+}{dt} = -iK_j\psi_j^+ + iU_j^{(3)}\psi_j\psi_j^{+2} + \sqrt{iU_j^{(3)}}\psi_j^+\zeta_j^+(t)$$

$\zeta_i(t)$ is a real, stochastic delta-correlated Gaussian noise, from nonlinearities:

$$\left\langle\zeta_i(t,x)\zeta_j(t',x')\right\rangle = \delta_{ij}\delta\left(t-t'\right)\delta\left(x-x'\right)$$
$$\left\langle\zeta_i^+(t,x)\zeta_j^+(t',x')\right\rangle = \delta_{ij}\delta\left(t-t'\right)\delta\left(x-x'\right).$$

+P equations in an optical lattice

Single mode case of an anharmonic oscillator

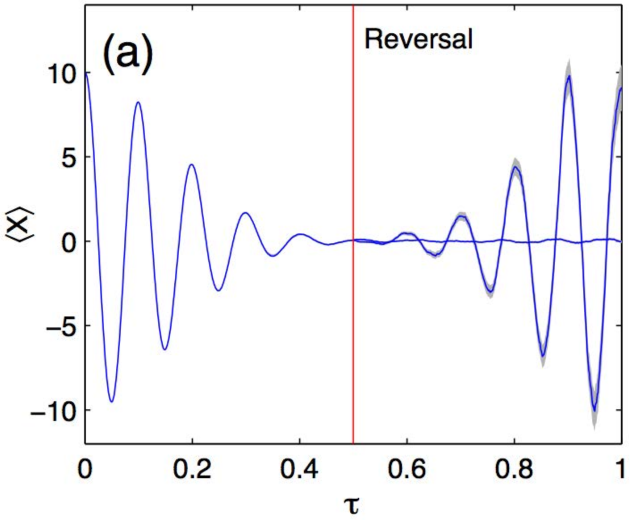
$$\frac{d\alpha}{dt} = -i\chi\alpha^2\beta + \sqrt{-i\chi}\alpha\zeta_1(t)$$
$$\frac{d\beta}{dt} = i\chi\beta^2\alpha + \sqrt{i\chi}\beta\zeta_2(t)$$

$\zeta_i(t)$ is a real, stochastic delta-correlated Gaussian noise:

$$\langle\zeta_i(t)\zeta_j(t')\rangle = \delta_{ij}\delta\left(t - t'\right).$$

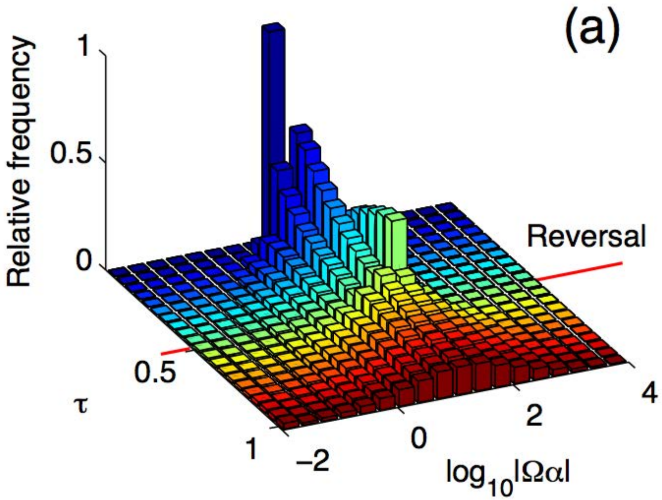
- What happens if we change the sign of χ?
- This is the same as reversing the time-direction.
- How can stochastic processes be reversible?

Time-reversal test: up to 10^{23} interacting bosons



Graph is for 100 photon case

Phase-space distribution is not unique!



Initial and final quantum states identical, distributions have changed!

+P equations in a parametric waveguide

Single mode case of an anharmonic oscillator

$$\frac{d\psi_0}{dt} = -\gamma_0\psi_0 + \mathscr{E}(x) - \chi\psi_1\psi_2 + \frac{iv_0^2}{2\omega_0}\nabla^2\psi_0$$
$$\frac{d\psi_1}{dt} = -\gamma_1\psi_1 + \chi\psi_0\psi_2 + \frac{iv_1^2}{2\omega_1}\nabla^2\psi_1 + \sqrt{\chi\psi_0}\xi_1$$
$$\frac{d\psi_2}{dt} = -\gamma_2\psi_2 + \chi\psi_0\psi_1 + \frac{iv_2^2}{2\omega_2}\nabla^2\psi_2 + \sqrt{\chi\psi_0}\xi_2$$

$\zeta_i(t)$ is a complex, stochastic delta-correlated Gaussian noise:

$$\left\langle \zeta_i(\mathbf{x},t)\zeta_j(\mathbf{x}',t')\right\rangle = \delta_{ij}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right).$$

Weighted stochastic gauge equations

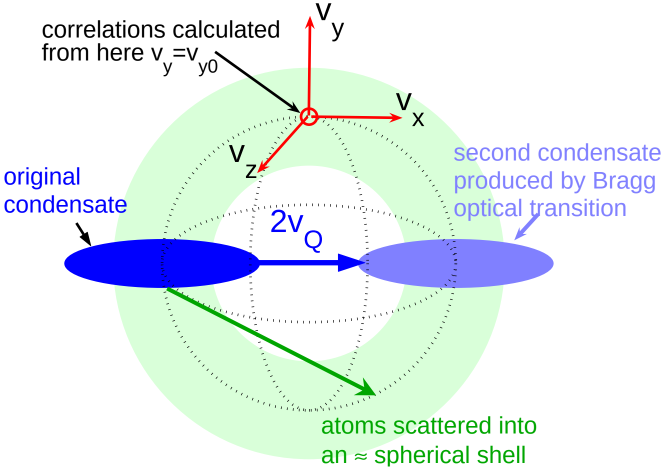
Hard quantum problems → tractable stochastic equations

Can improve sampling error using a weight factor Ω

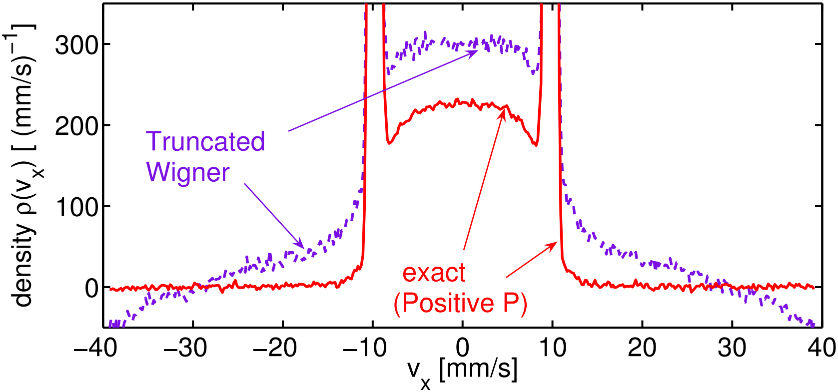
$$d\Omega/\partial t = \Omega \mathbf{g} \cdot \boldsymbol{\zeta}$$
$$d\boldsymbol{\alpha}/\partial t = \mathbf{A} + \mathbf{B}(\boldsymbol{\zeta} - \mathbf{g})$$

- Can be used for fermions OR bosons
  - Many trajectories needed to control growing sampling errors
  - **g is a gauge chosen to stabilize trajectories**
  - A careful choice of basis, gauge and stochastic method is necessary

BEC collision: 10^5 bosons, 10^6 spatial modes



Positive-P vs Truncated Wigner



3D Truncated Wigner: diverges, too few particles per mode!
+P: converges, but the sampling error increases with time

SUMMARY

Phase-space representation methods have many applications

Phase-space approach is relatively simple!

- Maps **quantum field evolution** into a stochastic equation
- Can also be used to treat interferometry
- Advantage: No exponential complexity issues!
- Mathematical challenge:
  - truncation error for Wigner methods
  - sampling error can grow with time

SUMMARY

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