IV. Theory of Quantum Measurements

The standard quantum theory governed by Schrödinger (or Heisenberg) equation describes the evolution of a system with two unique features. The evolution described by the standard quantum theory is a deterministic and reversible process. If the initial state of a system is known and a Hamiltonian is given, the final state is uniquely determined by the unitary operator $\hat{U}$. We can always undo this time evolution by imposing the inverse unitary operator $\hat{U}^{-1}$ and recover the initial state. On the other hand, a process of quantum measurement is non-deterministic and irreversible. Even though we have a perfect information about a system, a measurement result is generally random and unpredictable. This is not due to the detector noise but rather due to the intrinsic uncertainty of the system. Once a measurement is completed and a result is read out, we cannot go back to the initial state. We even do not know what the initial state was. In this Chapter, we present the theory of a quantum measurement process. We start with the von Neumann's projection postulate and extend it to an approximate measurement with a finite error. We then explain the most relevant concept for our purpose: the difference between linear and nonlinear continuous measurements. A quantum Zeno effect in continuous measurements and contextuality in a measurement-feedback system are mentioned as representative examples of nonlinear continuous measurements.

4.1 Exact measurements

Any quantum measurement process is fully characterized by answering the three following questions.

Q1: What is a measurement result?

A1: It is one of the eigenvalues q_n of a measured quantity, often called an observable $\hat{q}$, which is defined by

$$\hat{q}\left|q_n\right\rangle = q_n\left|q_n\right\rangle . \tag{1}$$

Q2: What is a probability $P\left(q_n\right)$ of obtaining a specific result q_n?

A2: It is computed with a projection operator $\left|q_n\right\rangle\left\langle q_n\right|$ and a system's density operator $\hat{\rho}$,

$$P\left(q_n\right) = \mathrm{Tr}\left(\left|q_n\right\rangle\left\langle q_n\right| \otimes \hat{\rho}\right) , \tag{2}$$

where Tr stands for a trace operation.

Q3: What is a post-measurement state $\hat{\rho}\left(q_n\right)$?

A3: It is given by

$$\hat{\rho}\left(q_n\right) = \frac{1}{P\left(q_n\right)} \left|q_n\right\rangle \left\langle q_n\right| \hat{\rho} \left| q_n\right\rangle \left\langle q_n\right| . \tag{3}$$

The reason why we use the expression (Eq. (3)) rather than $\hat{\rho}\left(q_n\right) = \left|q_n\right\rangle \left\langle q_n\right|$ is its generality for a compound system. For instance, if an initial state of a bipartite system is

$$\hat{\rho} = c_1 \left|q_1\right\rangle_{A\,A} \left\langle q_1\right| \otimes \left|p_1\right\rangle_{B\,B} \left\langle p_1\right| + c_2 \left|q_2\right\rangle_{A\,A} \left\langle q_2\right| \otimes \left|p_2\right\rangle_{B\,B} \left\langle p_2\right| , \tag{4}$$

the probability of measuring q_1 for a subsystem A is c_1. If the measurement result for a subsystem A is indeed q_1, the final state should be

$$\hat{\rho}\left(q_1\right) = \left|q_1\right\rangle_{A\,A} \left\langle q_1\right| \otimes \left|p_1\right\rangle_{B\,B} \left\langle p_1\right| . \tag{5}$$

The above von Neumann's recipe is not very useful for our purpose, because it describes an exact measurement without an error. Such an ideal measurement is difficult to realize in actual experiments. We need to extend the von Neumann's recipe to include a finite measurement error.

4.2 Approximate measurements

We now introduce an indirect measurement model, shown in Fig. 1, to describe an approximate measurement [1]. We aim to measure an observable $\hat{q}_s$ of a system using a readout observable $\hat{P}_p$ of a probe. In the first step, we set up the interaction Hamiltonian between the system and the probe in order to transfer the information about the observable $\hat{q}_s$ of the system to the readout observable $\hat{P}_p$ of the probe. This is achieved by creating a quantum correlation between the two observables $\hat{q}_s$ and $\hat{P}_p$. In the second step, we switch-off the interaction Hamiltonian between the system and the probe and measure the readout observable $\hat{P}_p$ by a macroscopic meter. This second step is a destructive measurement in which the post-measurement state of the probe is completely unpredictable due to the disturbance injected from the macroscopic meter. However, the system is protected against this noise injection from the macroscopic meter, since the system and the probe are decoupled before the second step is switched-on.

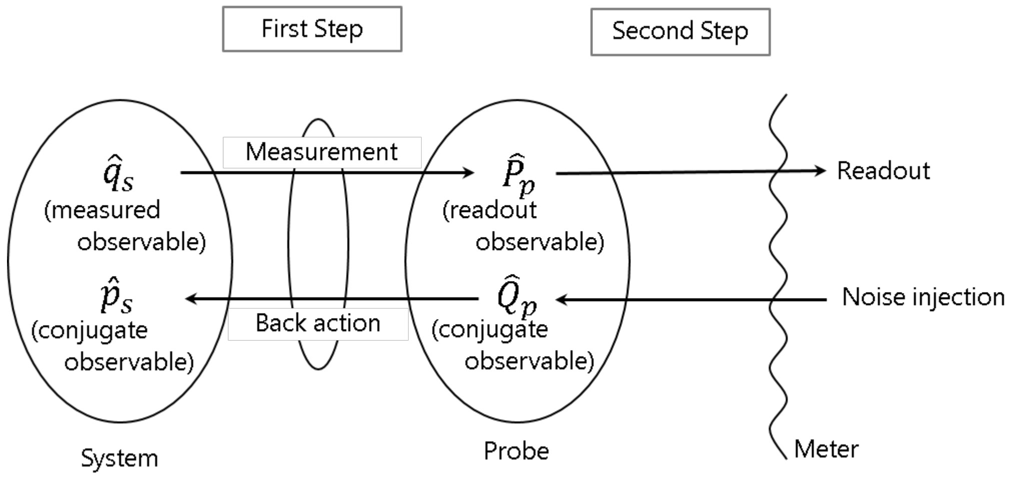

FIG. 1: A model for indirect quantum measurements.

4.2.1 Measurement error and back action noise

Let us consider a simple interaction Hamiltonian between the system and the probe:

$$\hat{\mathcal{H}}_{\mathrm{I}} = \hbar\chi\hat{q}_s \otimes \hat{Q}_p, \tag{6}$$

where $\hat{Q}_p$ is a conjugate observable to $\hat{P}_p$. The commutation relation for $\hat{Q}_p$ and $\hat{P}_p$ is defined by

$$\left[\hat{Q}_p, \hat{P}_p\right] = i\hbar. \tag{7}$$

Similarly, we have the commutation relation for $\hat{q}_s$ and $\hat{p}_s$:

$$\left[\hat{q}_s, \hat{p}_s\right] = i\hbar. \tag{8}$$

The Heisenberg equations of motion for $\hat{P}_p$ and $\hat{p}_s$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_p = \frac{1}{i\hbar}\left[\hat{P}_p, \hat{\mathcal{H}}_{\mathrm{I}}\right] = -\chi\hat{q}_s, \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_s = \frac{1}{i\hbar}\left[\hat{p}_s, \hat{\mathcal{H}}_{\mathrm{I}}\right] = -\chi\hat{Q}_p. \tag{10}$$

Equation (9) is a desired one, because the time evolution of $\hat{P}_p$ is governed by $\hat{q}_s$, so that the information about $\hat{q}_s$ is transferred to $\hat{P}_p$ by such a unitary evolution. Equation (9) is time-integrated easily if we neglect the time evolution of $\hat{q}_s$ and we obtain

$$\hat{P}_p(t) - \hat{P}_p(0) = -\chi t \hat{q}_s(0). \tag{11}$$

We now define the inferred observable $\hat{q}_{s,\,\mathrm{obs}}$ by

$$\hat{q}_{s,\,\mathrm{obs}} \equiv \frac{\hat{P}_p(t)}{(-\chi t)} = \hat{q}_s(0) + \frac{\hat{P}_p(0)}{(-\chi t)}. \tag{12}$$

When we prepare the probe in a way, $\left\langle \hat{P}_p(0) \right\rangle = 0$, the expectation value for the inferred observable $\hat{q}_{s,\,\mathrm{obs}}$ is identical to the expectation value of the measured observable $\hat{q}_s$:

$$\langle \hat{q}_{s,\,\mathrm{obs}} \rangle = \langle \hat{q}_s(0) \rangle. \tag{13}$$

This is called "no bias condition". Even though we can easily realize the no bias condition, $\left\langle \hat{P}_p(0) \right\rangle = 0$, the variance of $\hat{P}_p(0)$ is generally non-zero. We can assume $\hat{q}_s(0)$ and $\hat{P}_p(0)$ are uncorrelated, i.e. no information transfer is achieved between the signal and the probe before the measurement. Then, the variance of the inferred observable $\hat{q}_{s,\,\mathrm{obs}}$ is given as the sum of two terms:

$$\left\langle \triangle \hat{q}_{s,\,\mathrm{obs}}^2 \right\rangle = \left\langle \triangle \hat{q}_s(0)^2 \right\rangle + \frac{\left\langle \triangle \hat{P}_p(0)^2 \right\rangle}{(\chi t)^2}. \tag{14}$$

The first term is an intrinsic uncertainty that the system has before the measurement, while the second term is an extrinsic uncertainty (measurement error) imposed by the probe. We can reduce a measurement error by decreasing the variance $\left\langle \triangle \hat{P}_p(0)^2 \right\rangle$ or increasing the coupling strength χt between the system and the probe.

Equation (10) describes an unavoidable disturbance imposed on the system by the quantum measurement process. Equation (10) is easily time-integrated if we neglect the time evolution of $\hat{Q}_p$ and we obtain

$$\hat{p}_s(t) - \hat{p}_s(0) = -\chi t \hat{Q}_p(0). \tag{15}$$

Since we can assume $\hat{p}_s(0)$ and $\hat{Q}_p(0)$ are uncorrelated, the variance of the conjugate observable $\hat{p}_s(t)$ is given as the sum of two terms:

$$\left\langle \triangle\hat{p}_s(t)^2 \right\rangle = \left\langle \triangle\hat{p}_s(0)^2 \right\rangle + (\chi t)^2 \left\langle \triangle\hat{Q}_p(0)^2 \right\rangle . \tag{16}$$

The first term is an intrinsic uncertainty that the system has before the measurement, while the second term is an extrinsic uncertainty (back action noise) imposed by the probe. We can reduce the back action noise by decreasing the variance $\left\langle \triangle\hat{Q}_p(0)^2 \right\rangle$ or decreasing the coupling strength χt.

If the probe is prepared in a minimum uncertainty wavepacket, $\left\langle \triangle\hat{Q}_p(0)^2 \right\rangle \left\langle \triangle\hat{P}_p(0)^2 \right\rangle = \hbar^2/4$, the product of the measurement error and the back action noise takes its minimum possible value:

$$\left\langle \triangle\hat{q}^2_{\text{meas, error}} \right\rangle \left\langle \triangle\hat{p}^2_{\text{back action}} \right\rangle = \frac{\hbar^2}{4}, \tag{17}$$

where $\triangle\hat{q}_{\text{meas, error}} = \hat{P}_p(0)/(-\chi t)$ and $\triangle\hat{p}_{\text{back action}} = (-\chi t)\,\hat{Q}_p(0)$. Note that the Heisenberg uncertainty principle imposes two independent constraints in quantum measurements. It imposes the limitation on how accurately we can prepare the (measured) system. It also imposes the independent limitation on how accurately we can prepare the (measuring) probe. In Chapter I we described that the quantum neural networks (QNN) operate very closely in this Heisenberg limit: both measured system (DOPO squeezed state) and measuring probe (incident vacuum state from the open port of the output coupler) are prepared in minimum uncertainty wavepackets.

4.2.2 Measurement probability and post-measurement state

In indirect quantum measurements, the system and the probe are initially independent so that the total density operator is a product state,

$$\hat{\rho}_i = \hat{\rho}_s \otimes \hat{\rho}_p. \tag{18}$$

The interaction Hamiltonian implements a unitary evolution $\hat{U}$, which brings the system and the probe to a joint-correlated state,

$$\hat{\rho}_f = \hat{U}\hat{\rho}_s \otimes \hat{\rho}_p\hat{U}^+, \tag{19}$$

where $\hat{U} = \exp\left(\hat{\mathcal{H}}_I t/i\hbar\right)$. After switching-off the interaction Hamiltonian $\hat{\mathcal{H}}_I$, the probe is coupled to the macroscopic meter. The probe density operator is then diagonalized in a "pointer-basis" $\left|P\right\rangle_p$ by the second step [2]. The choice of the pointer-basis is decided by an experimenter. In our case, the macroscopic meter is an optical homodyne detector with a high-intensity local oscillator field and square-law photodetector. The chosen pointer-basis is a quadrature amplitude eigenstate. Note that this second step is an exact measurement of $\hat{P}_p$.

We now introduce an inferred value $\tilde{q}$ instead of an actual measurement result P defined by

$$\tilde{q} = {}^{P}\!/_{(-\chi t)}. \tag{20}$$

The von Neumann's projection operator for the second step is thus given by $\left|\tilde{q}\right\rangle_{p\,p}\left\langle\tilde{q}\right|$. The probability of obtaining a specific result $\tilde{q}$ is obtained as

$$\begin{aligned} P\left(\tilde{q}\right) &= \mathrm{Tr}_p\left[\left|\tilde{q}\right\rangle_{p\,p}\left\langle\tilde{q}\right|\hat{\rho}_p^{(\mathrm{red})}\right] \\ &= \mathrm{Tr}_s\mathrm{Tr}_p\left[\left|\tilde{q}\right\rangle_{p\,p}\left\langle\tilde{q}\right|\hat{U}\hat{\rho}_s\otimes\hat{\rho}_p\hat{U}^+\right] \\ &= \mathrm{Tr}_s\left[\hat{X}\left(\tilde{q}\right)\hat{\rho}_s\right]. \end{aligned} \tag{21}$$

Here $\hat{\rho}_p^{(\mathrm{red})} = \mathrm{Tr}_s\left[\hat{U}\rho_s\otimes\hat{\rho}_p\hat{U}^+\right]$ is the reduced density operator of the probe after the first step of the unitary evolution and $\hat{X}\left(\tilde{q}\right) = \mathrm{Tr}_p\left[\hat{U}^+\left|\tilde{q}\right\rangle_{p\,p}\left\langle\tilde{q}\right|\hat{U}\hat{\rho}_p\right]$ is the generalized projection operator. $\hat{X}\left(\tilde{q}\right)$ identifies the three steps of indirect measurements: how to prepare the probe $(\hat{\rho}_p)$, how to transfer the information from the system to the probe $\left(\hat{U}\right)$ and what is the actual measurement result $\left(\left|\tilde{q}\right\rangle_{p\,p}\left\langle\tilde{q}\right|\right)$.

The state is computed by projecting a probe state $\left|\tilde{q}\right\rangle_p$, which corresponds to the measurement result, onto the joint-correlated state $\hat{\rho}_f$ and normalizing by the measurement probability $P\left(\tilde{q}\right)$:

$$\begin{aligned} \hat{\rho}_s\left(\tilde{q}\right) &= \frac{1}{P\left(\tilde{q}\right)}\,{}_p\left\langle\tilde{q}\right|\hat{\rho}_f\left|\tilde{q}\right\rangle_p \\ &= \frac{1}{P\left(\tilde{q}\right)}\hat{M}\left(\tilde{q}\right)\hat{\rho}_s\hat{M}\left(\tilde{q}\right)^+, \end{aligned} \tag{22}$$

where $\hat{M}\left(\tilde{q}\right) = {}_p\left\langle\tilde{q}\right|\hat{U}\left|\psi\right\rangle_p$ is the operator amplitude for the probe to evolve from its initial state $|\psi\rangle_p$ to the final state $|\tilde{q}\rangle_p$ via the unitary operator $\hat{U}$, and fully describes the three steps of indirect measurements [1].

If we substitute $\hat{\rho}_p = |\psi\rangle_{p\,p}\langle\psi|$ (pure state) into the above expression for $\hat{X}(\tilde{q})$, we obtain

$$\begin{aligned}\hat{X}(\tilde{q}) &= \mathrm{Tr}_p\left[\hat{U}^+ |\tilde{q}\rangle_{p\,p}\langle\tilde{q}|\,\hat{U}\,|\psi\rangle_{p\,p}\langle\psi|\right] \\ &= \hat{M}^+(\tilde{q})\,\hat{M}(\tilde{q})\,.\end{aligned} \tag{23}$$

We can further substitute an identity operator, $\hat{I} = \int |q\rangle_{s\,s}\langle q|\,dq$, between $\hat{M}^+(\tilde{q})$ and $\hat{M}(\tilde{q})$ in Eq.(23) to obtain

$$\begin{aligned}\hat{X}(\tilde{q}) &= \int \hat{M}^+(\tilde{q})\,|q\rangle_{s\,s}\langle q|\,\hat{M}(\tilde{q})\,\mathrm{d}q \\ &= \int |q\rangle_s\, x(\tilde{q},q)\ {}_s\langle q|\,\mathrm{d}q,\end{aligned} \tag{24}$$

where

$$\begin{aligned}x(\tilde{q},q) &=_p \langle\psi|\,\hat{U}^+(q)\,|\tilde{q}\rangle_{p\,p}\langle\tilde{q}|\,\hat{U}(q)\,|\psi\rangle_p \\ &= \left|{}_p\langle\tilde{q}|\,\hat{U}(q)\,|\psi\rangle_p\right|^2.\end{aligned} \tag{25}$$

This is the conditional probability that the macroscopic meter reports a specific result $\tilde{q}$ when the system was in an eigenstate $|q\rangle_s$ before the measurement.

The generalized projection operator $\hat{X}(\tilde{q})$ has the non-negative eigenvalues,

$${}_s\langle q|\,\hat{X}(\tilde{q})\,|q\rangle_s = x(\tilde{q},q) \geq 0, \tag{26}$$

so that it is a positive operator. $\hat{X}(\tilde{q})$ is also the decomposition of unity:

$$\begin{aligned}\int \hat{X}(\tilde{q})\,\mathrm{d}\tilde{q} &= \int |q\rangle_s \int_{-\infty}^{\infty} x(\tilde{q},q)\,\mathrm{d}\tilde{q}\ {}_s\langle q|\,\mathrm{d}q \\ &= \int |q\rangle_{s\,s}\langle q|\,\mathrm{d}q \\ &= \hat{I}.\end{aligned} \tag{27}$$

In general, a positive operator which is the decomposition of unity describes a physically realizable quantum measurement and is called a positive operator valued measure (POVM). However, it should be noted that a POVM measurement is a highly mathematical concept and we do not know often how to construct an actual detector for a given POVM.

4.2.3. Optical homodyne detection

A particular quantum measurement device used in the measurement-feedback quantum neural network (MF-QNN) is an optical homodyne detector shown in Fig. 2. A part of the degenerate optical parametric oscillator (DOPO) pulse is picked-off by an output coupler and combined with a high-intensity local oscillator pulse with a 50-50% beam splitter. The two output fields of the 50-50% beam splitter are fed into the square-law photodetectors, and then the two output currents are input into a subtraction circuit. This particular configuration is called balanced homodyne detectors and can measure a single quadrature amplitude of the out-coupled field exactly.

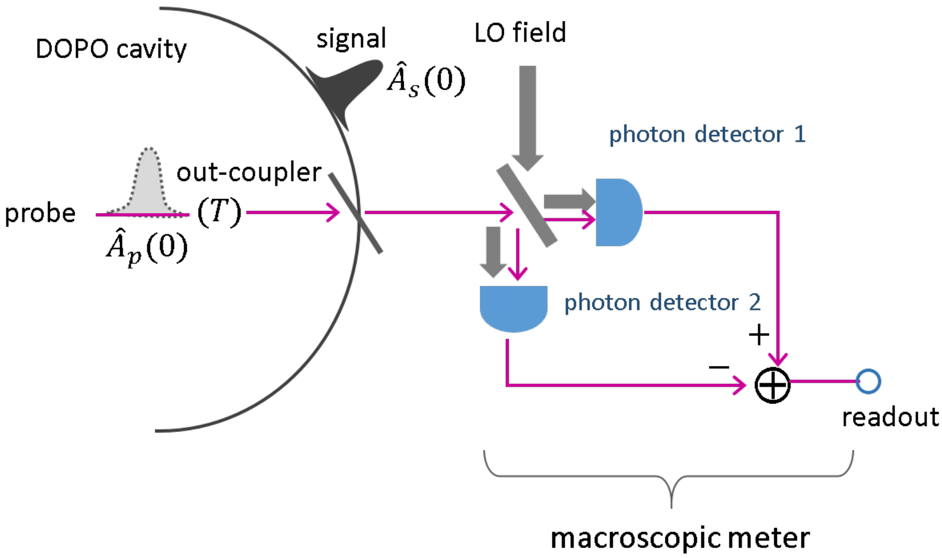

FIG. 2: An optical balanced homodyne detector.

Let us compute the measurement error and the back action noise of this particular detector. The out-coupled field (probe) is given by

$$\hat{A}_p(t) = \sqrt{1-T}\hat{A}_s(0) + \sqrt{T}\hat{A}_p(0), \tag{28}$$

where T is the (power) transmission coefficient of the out-coupler, $\hat{A}_s(0)$ is the signal input field operator and $\hat{A}_p(0)$ is the probe input field operator. The inferred observable defined by Eq.(12) is obtained as

$$\hat{A}_{s,\,\mathrm{obs}} \equiv \frac{\hat{A}_p(t)}{\sqrt{1-T}} = \hat{A}_s(0) + \sqrt{\frac{T}{1-T}}\hat{A}_p(0), \tag{29}$$

or more directly

$$\hat{X}_{s,\,\mathrm{obs}} = \hat{X}_s\,(0) + \sqrt{\frac{T}{1-T}}\hat{X}_p\,(0)\,. \tag{30}$$

Here $\hat{X}_i = {}^1\!/_2 \left(\hat{A}_i + \hat{A}_i^+\right)$ is the in-phase amplitude operator. Since the probe input field is in a vacuum state in our case, the "no bias condition" is satisfied:

$$\left\langle \hat{X}_{s,\,\mathrm{obs}} \right\rangle = \left\langle \hat{X}_s\,(0) \right\rangle. \tag{31}$$

The total variance of the inferred observable consists of the intrinsic uncertainty of the signal input field and the extrinsic measurement error:

$$\left\langle \triangle\hat{X}^2_{s,\,\mathrm{obs}} \right\rangle = \left\langle \triangle\hat{X}_s\,(0)^2 \right\rangle + \left(\frac{T}{1-T}\right)\left\langle \triangle\hat{X}_p\,(0)^2 \right\rangle. \tag{32}$$

The internal signal field after the out-coupler is given by

$$\hat{A}_s\,(t) = \sqrt{T}\hat{A}_s\,(0) - \sqrt{1-T}\widehat{A}_p\,(0)\,. \tag{33}$$

We can assume that the loss of the signal field is compensated for by a noise-less phase sensitive amplifier with a gain factor $G = {}^1\!/_T$, which leads to the overall quadrature-phase amplitude of the signal field after the measurement and amplification:

$$\hat{P}_s\,(t) = \hat{P}_s\,(0) - \sqrt{\frac{1-T}{T}}\hat{P}_p\,(0)\,, \tag{34}$$

where $\hat{P}_i = \left(\hat{A}_i - \hat{A}_i^+\right)/2i\;(i = s,\, p)$ is the quadrature-phase amplitude. The variance of the conjugate observable $\hat{P}_s\,(t)$ is thus the sum of the original uncertainty and the measurement back action noise:

$$\left\langle \triangle\hat{P}_s\,(t)^2 \right\rangle = \left\langle \triangle\hat{P}_s\,(0)^2 \right\rangle + \left(\frac{1-T}{T}\right)\left\langle \triangle\hat{P}_p\,(0)^2 \right\rangle. \tag{35}$$

The input probe state is a vacuum state with $\left\langle \triangle\hat{X}_p\,(0)^2 \right\rangle = \left\langle \triangle\hat{P}_p\,(0)^2 \right\rangle = {}^1\!/_4$, and thus the product of the measurement error, $({}^T\!/_{1-T})\left\langle \triangle\hat{X}_p\,(0)^2 \right\rangle$, and the measurement back action noise, $({}^{1-T}\!/_T)\left\langle \triangle\hat{P}_p\,(0)^2 \right\rangle$, satisfies the minimum uncertainty product. In this sense, the optical homodyne detection is an ideal indirect quantum measurement.

4.3 Continuous measurements

So far, we have studied a discrete measurement, in which the system and probe are prepared in particular initial states, then the joint-correlated states are produced by switching-on the interaction Hamiltonian between the two, and finally the probe is destructively and exactly measured by the macroscopic meter. However, in real physical measurements, we often encounter a different situation. For instance, in Fig. 3, an unknown external force $F(t)$ couples to the measured observable $\hat{x}_s(t)$ of the system. We wish to monitor continuously $\hat{x}_s(t)$ by using the readout observable $\hat{X}_p(t)$ of the probe to detect not only arrival of the unknown force but also the time-dependent force shape $F(t)$. In such a continuous measurement, there emerge two kinds of back action noise: fluctuational back action noise and dynamic back action noise. The fluctuational back action noise is a random and unpredictable perturbation imposed on the uncertainty $\langle \triangle \hat{x}_s^2 \rangle$ of the measured observable, while the dynamic back action noise is a regular and predictable influence on the evolution of the expectation value $\langle \hat{x}_s(t) \rangle$ of the measured observable. If the above two are independent so that the dynamic back action noise can be eliminated, in principle, such a measurement is called a "linear continuous measurement". If the above two back action noise are not separable so that the dynamic back action noise is unavoidable, such a measurement is called a "nonlinear continuous measurement". We will describe the difference of these two measurements in the section (4.5).

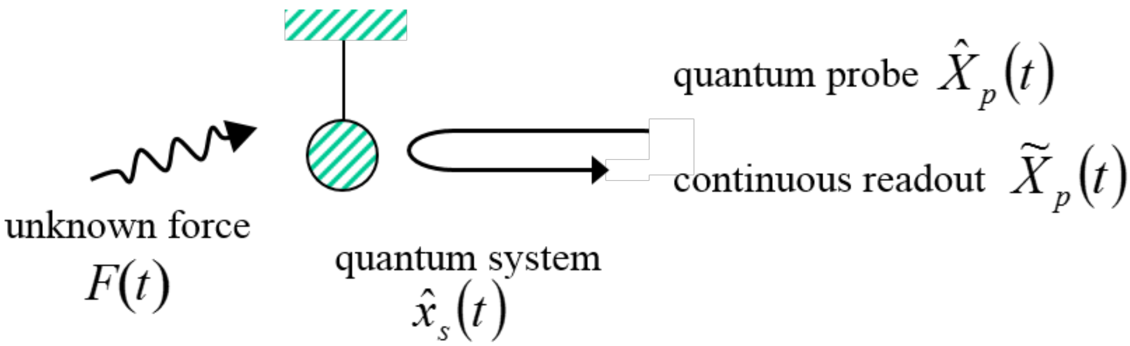

FIG. 3: A model for continuous quantum measurements.

Let us consider n consecutive measurements of $\hat{x}_s(t)$ over a time interval τ. Each measurement is performed over a small interval $\theta = \tau/n$. In general, the inferred observable $\tilde{x}(t)$ for each measurement is expressed as

$$\tilde{x}(t) = \hat{x}_s(t) + \triangle\hat{x}_\theta(t), \tag{36}$$

where $\triangle\hat{x}_\theta(t)$ represents the measurement error governed by the internal noise of the probe for each measurement performed for an interval θ. If such discrete measurements are repeated n times over an interval τ, the overall measurement error is given by

$$\left\langle \triangle\hat{x}_\tau^2 \right\rangle = \frac{\langle \triangle\hat{x}_\theta^2 \rangle}{n} = \frac{\langle \triangle\hat{x}_\theta^2 \rangle \theta}{\tau} \underset{(\theta\to 0)}{\to} \frac{S_x}{\tau}, \tag{37}$$

where $S_x = \lim_{\theta\to 0} \langle \triangle\hat{x}_\theta^2 \rangle \theta$ is the spectral density of $\triangle\hat{x}_\theta$. Equation (37) is a familiar result to us, since the measurement noise power associated with a measurement time interval τ is reduced by increasing τ (or decreasing the bandwidth $B = 1/\tau$).

The conjugate observable $\hat{p}_s(t)$ of the system after each measurement is expressed as

$$\tilde{p}(t) = \hat{p}_s(t) + \triangle\hat{p}_\theta(t), \tag{38}$$

where $\triangle\hat{p}_\theta(t)$ represents the back action noise also governed by the internal noise of the probe for each measurement performed for an interval θ. Each back action noise is independent for n repeated measurements, since the probe is prepared independently for all measurements. Thus, the overall back action noise for n repeated measurements is given by

$$\left\langle \triangle\hat{p}_\tau^2 \right\rangle = \left\langle \triangle\hat{p}_\theta^2 \right\rangle n = \frac{\langle \triangle\hat{p}_\theta^2 \rangle \tau}{\theta} \underset{(\theta\to 0)}{\to} S_p \cdot \tau, \tag{39}$$

where $S_p = \lim_{\theta\to 0} \langle \triangle\hat{p}_\theta^2 \rangle / \theta$ is the spectral density of $\triangle\hat{p}_\theta$. The back action noise power associated with a measurement time interval τ is increased by increasing τ (or decreasing the bandwidth), which is also a reasonable result.

If we can prepare the probe for each measurement in a minimum uncertainty wavepacket, the two spectral densities satisfy the Heisenberg limit:

$$S_x \cdot S_p = \left\langle \triangle\hat{x}_\theta^2 \right\rangle \left\langle \triangle\hat{p}_\theta^2 \right\rangle = \frac{\hbar^2}{4}. \tag{40}$$

Figure 4 shows the time evolution of the system, which is driven by an external unknown force $F(t)$ and simultaneously monitored by the probe. The initial density operator $\hat{\rho}_{\text{in}}$ of the system is translated by the unitary operator $V_1 = \exp\left(\hat{\mathcal{H}}_s \tau_1 / i\hbar\right)$, where $\hat{\mathcal{H}}_s$ describes the interaction between the system and external unknown force such as,

$$\hat{\mathcal{H}}_s = \frac{\hat{p}_s^2}{2m} + \frac{1}{2}k\hat{x}_s^2 - \hat{x}_s\mathcal{F}(t). \tag{41}$$

The post unitary evolution state is given by

$$\hat{\rho}_1 = \hat{V}_1\hat{\rho}_{\text{in}}\hat{V}_1^\dagger. \tag{42}$$

The three steps in the quantum measurement, i.e. the preparation of the probe, the information transfer from the system to the probe and the readout of the probe coordinate, are described by the operator amplitude $\hat{M}(\tilde{q}_1) =_p \langle\tilde{q}|\,\hat{U}\,|\psi_1\rangle_p$. The post-measurement state is then given by

$$\begin{aligned}\hat{\rho}_1' &= \frac{1}{P(\tilde{q}_1)}\hat{M}(\tilde{q}_1)\,\hat{\rho}_1\hat{M}(\tilde{q}_1)^\dagger \\ &= \frac{1}{P(\tilde{q}_1)}\hat{M}(\tilde{q}_1)\,\hat{V}_1\hat{\rho}_{\text{in}}\hat{V}_1^\dagger\hat{M}(\tilde{q}_1)^\dagger.\end{aligned} \tag{43}$$

If this process is repeated n times, the final density operator is expressed as

$$\hat{\rho}_n' = \frac{1}{P(\tilde{q}_1)\,P(\tilde{q}_2)\cdots P(\tilde{q}_n)}\hat{Y}(\tilde{q}_1,\,\tilde{q}_2,\,\cdots\,\tilde{q}_n)\,\hat{\rho}_{\text{in}}\hat{Y}(\tilde{q}_1,\,\tilde{q}_2,\,\cdots\,\tilde{q}_n)^\dagger, \tag{44}$$

where

$$\hat{Y}(\tilde{q}_1,\,\tilde{q}_2,\,\cdots\,\tilde{q}_n) = \hat{M}(\tilde{q}_n)\,\hat{V}_n\cdots\cdots\hat{M}(\tilde{q}_1)\,\hat{V}_1. \tag{45}$$

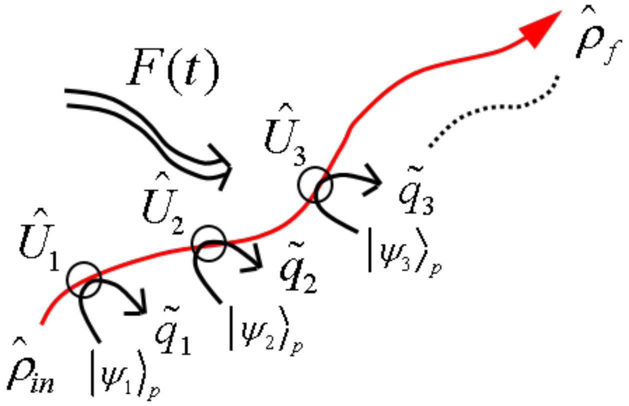

FIG. 4: A model for continuous measurements.

Equation (44) is the key equation for describing the dynamics of measurement-feedback quantum neural networks (MF-QNN), in which the unitary operator $\hat{V}$ represents the feedback signal injection as well as parametric amplification and linear loss rather than the external unknown force coupling (see the next chapter in detail).

4.4 Non-referred measurements

If we ask the dynamics of a whole ensemble of the systems, irrespective of the measurement results as shown in Fig. 5, the relevant density operator is expressed by

$$\begin{aligned}\hat{\rho}_s^{(\mathrm{red})} &= \sum_{\tilde{q}} P(\tilde{q})\,\hat{\rho}_s(\tilde{q}) \\ &= \sum_{\tilde{q}} {}_p\langle\tilde{q}|\,\hat{U}\hat{\rho}_s \otimes \hat{\rho}_p \hat{U}^\dagger\,|\tilde{q}\rangle_p \\ &= \mathrm{Tr_p}\left(\hat{U}\hat{\rho}_s \otimes \hat{\rho}\hat{U}^\dagger\right),\end{aligned} \tag{46}$$

where $P(\tilde{q})$ is the probability of obtaining a specific result $\tilde{q}$ and $\hat{\rho}_s(\tilde{q})$ is the post-measurement state when the measurement result is $\tilde{q}$. Equation (46) shows the expected result that a non-referred measurement is equivalent to a simple dissipation process, in which the probe information is simply lost to the external reservoir. If the probe is prepared in a pure state $|\psi\rangle_p$, the evolution of the density operator is reduced to

$$\hat{\rho}_s^{(\mathrm{red})} = \sum_{\tilde{q}} \hat{M}(\tilde{q})\,\hat{\rho}_s \hat{M}(\tilde{q})^\dagger . \tag{47}$$

Let us consider the evolution of the system under continuous but non-referred measurement. The total Hamiltonian for a small time interval is given by

$$\hat{\mathcal{H}_T} = \hat{\mathcal{H}}_j - \hat{q}_s \otimes \hat{Q}_p, \tag{48}$$

where $\hat{\mathcal{H}}_j$ is the free Hamiltonian of the system for a time interval j , $\hat{q}_s$ and $\hat{Q}_p$ are the measured observable of the system and the conjugate observable of the readout observable of the probe, respectively (see Fig. 1). The unitary evolution operator is approximated for a small time interval θ:

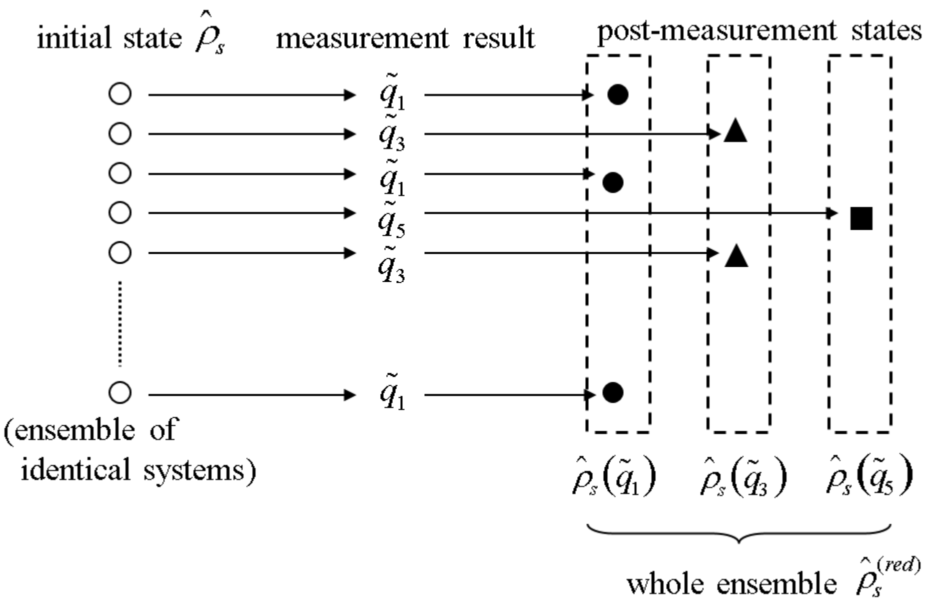

FIG. 5: A non-referred measurement.

$$\begin{aligned}\hat{U}_j &= \exp\left(\frac{\hat{\mathcal{H}}_T}{i\hbar}\theta\right) \\ &\simeq 1 + \frac{\theta}{i\hbar}\hat{\mathcal{H}}_T - \frac{\theta^2}{2\hbar^2}\hat{\mathcal{H}}_T^2.\end{aligned} \tag{49}$$

The system density operator $\hat{\rho}_{j+1}$ at a time $j+1$ is related to the system density operator ρ_j at a time j by

$$\begin{aligned}\hat{\rho}_{j+1} &= \mathrm{Tr}_p\left[\hat{U}_j \left|\psi_j\right\rangle_p \hat{\rho}_{j\,p} \left\langle\psi_j\right| \hat{U}_j^+\right] \\ &\simeq \mathrm{Tr}_p\left\{\left|\psi_j\right\rangle_p \hat{\rho}_{j\,p}\left\langle\psi_j\right| + \frac{\theta}{i\hbar}\left[\hat{\mathcal{H}}_T, \left|\psi_j\right\rangle_p \hat{\rho}_{j\,p}\left\langle\psi_j\right|\right] - \frac{\theta^2}{2\hbar^2}\left[\hat{\mathcal{H}}_T, \left[\hat{\mathcal{H}}_T, \left|\psi_j\right\rangle_p \hat{\rho}_{j\,p}\left\langle\psi_j\right|\right]\right]\right\} \\ &= \rho_j + \frac{\theta}{i\hbar}\left[\hat{\mathcal{H}}_j - \hat{q}_s\left\langle\hat{Q}_p\right\rangle, \hat{\rho}_j\right] \\ &- \frac{\theta^2}{2\hbar^2}\left\{\left[\hat{\mathcal{H}}_j, \left[\hat{\mathcal{H}}_j, \hat{\rho}_j\right]\right] - \left\langle\hat{Q}_p\right\rangle\left[\left[\hat{\mathcal{H}}_j, \left[\hat{q}_s, \hat{\rho}_j\right]\right] + \left[\hat{q}_s, \left[\hat{\mathcal{H}}_j, \hat{q}_s\right]\right] + \left\langle\hat{Q}_p^2\right\rangle\left[\hat{q}_s, \left[\hat{q}_s, \hat{\rho}_j\right]\right]\right]\right\}.\end{aligned} \tag{50}$$

Without loss of generality, we can prepare the initial probe state to satisfy $\left\langle\hat{Q}_p\right\rangle = 0$. The time evolution of the system density operator is then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \lim_{\theta\to 0}\frac{\hat{\rho}_{j+1} - \hat{\rho}_j}{\theta} = \frac{1}{i\hbar}\left[\hat{\mathcal{H}}, \hat{\rho}\right] - \frac{\sigma_F^2}{2\hbar^2}\left[\hat{q}_s, \left[\hat{q}_s, \hat{\rho}\right]\right], \tag{51}$$

where $\sigma_F^2 = \lim_{\theta \to 0} \left\langle \hat{Q}_p^2 \right\rangle$ is the strength of the back action noise and equivalent to the spectral density S_p defined by Eq.(39). In Chapter V, we will use Eq.(51) to describe the two important dissipation processes in MF-QNN: linear loss and nonlinear two photon loss.

4.5 Linear and nonlinear continuous measurements

A linear continuous measurement is a theoretical concept (approximation) for very weak measurement limit, in which the measurement error is so large that the system's free evolution is not influenced by the measurement process. In general, the system's free evolution is either weakly or strongly influenced by continuous measurements and , in some cases, the system is completely frozen in its initial state so that the standard unitary evolution is suppressed by the action of continuous measurements. We will discuss the two examples of such a nonlinear continuous measurement in this section.

4.5.1 Quantum Zeno effect

Let us consider a simple quantum optical system consisting of a single two-level atom and a single-mode cavity, as shown in Fig. 6. We introduce a Pauli spin operator to describe the combined system of a two-level atom and single-mode field within one excitation manifold,

$$\begin{aligned} |\uparrow\rangle &= |e\rangle_a |0\rangle_f , \\ |\downarrow\rangle &= |g\rangle_a |1\rangle_f . \end{aligned} \tag{52}$$

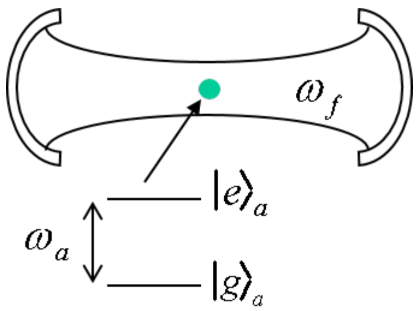

FIG. 6: A single two-level atom in a single-mode cavity.

Then, the interaction Hamiltonian between the atom and the field is given by

$$\hat{\mathcal{H}}_I = -\hbar g \hat{\sigma}_x , \tag{53}$$

where $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the standard Pauli operator. The unitary operator generated by this interaction Hamiltonian is

$$\hat{U} = \cos(gt)\,\hat{I} + i\,\sin(gt)\,\hat{\sigma}_x, \tag{54}$$

where $\hat{I}$ is the identity operator. When the initial state is

$$|\psi(0)\rangle = \cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} |\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} |\downarrow\rangle, \tag{55}$$

the final state $|\psi(t)\rangle = \hat{U}\,|\psi(0)\rangle$ has the probability of finding the $|\downarrow\rangle$ state as

$$\begin{aligned} P(\downarrow) &= \langle\psi(t)|\downarrow\rangle\langle\downarrow|\psi(t)\rangle \\ &= \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\sin^2(gt) + \frac{1}{2}\sin(\phi)\sin(\theta)\sin(2gt) \\ &= \begin{cases} \frac{1}{2}\left[1 \mp \cos(\Omega t)\right] & : \theta = 0 \,\text{or}\, \pi \\ \frac{1}{2} & : \theta = \frac{\pi}{2} \,\text{and}\, \phi = 0 \,\text{or}\, \pi \\ \frac{1}{2}\left[1 \pm \sin(\Omega t)\right] & : \theta = \frac{\pi}{2} \,\text{and}\, \phi = \frac{\pi}{2} \,\text{or}\, -\frac{\pi}{2}. \end{cases} \end{aligned} \tag{56}$$

Here, $\Omega = 2g$ is a vacuum Rabi frequency. The general solutions for the above vacuum Rabi oscillation for arbitrary initial states are schematically shown in Fig. 7. There is one-to-one correspondence between the initial state $|\psi(0)\rangle$ and the probability $P(\downarrow)$, except for the two-fold degeneracy of φ and $\pi - \varphi$.

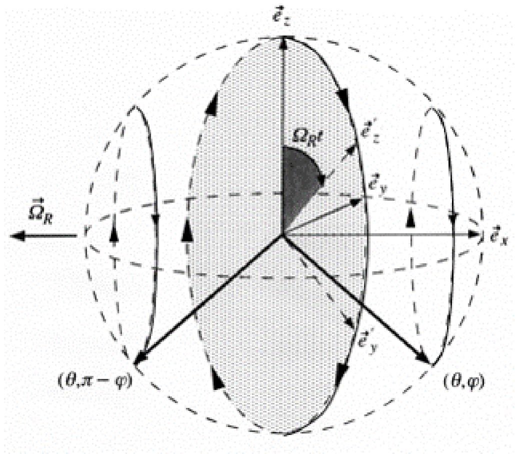

FIG. 7: A vacuum Rabi oscillation of the coupled single atom-cavity field with arbitrary initial states.[3]

Suppose the effective spin ($|\uparrow\rangle$ or $|\downarrow\rangle$) is monitored continuously by an external agent. The ensemble averaged density operator for such a case obeys Eq.(51). The measured observable is $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the fluctuational back action noise is represented by the parameter $\tau_0 = \left(\hbar/\sigma_F\right)^2$. If we substitute, $\hat{\rho} = 1/2\left(\hat{I} + \rho_x\hat{\sigma}_x + \rho_y\hat{\sigma}_y + \rho_z\hat{\sigma}_Z\right)$ into Eq.(51) and use the commutation relation, $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\hat{\sigma}_k \,(i,j,k = x,y,z)$, we obtain the so-called Bloch equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_x = -\frac{2}{\tau_0}\rho_x, \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_y = \Omega\rho_z - \frac{2}{\tau_0}\rho_y, \tag{58}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_z = -\Omega\rho_y. \tag{59}$$

The measurement effect is expressed by the parameter τ_0 .

The solution of Eqs.(57) - (59) with the initial conditions, $\rho_z(0) = 1$, $\rho_x(0) = \rho_y(0) = 0$, is given by

$$\rho_z(t) = \begin{cases} \left[\cos\left(\Omega' t\right) + \frac{1}{\Omega'\tau_0}\sin\left(\Omega' t\right)\right] e^{-\frac{t}{\tau_0}} & : \Omega\tau_0 > 1 \\ \frac{1}{\tau' - \tau''}\left(\tau' e^{-\frac{t}{\tau'}} - \tau'' e^{-\frac{t}{\tau''}}\right) & : 0 < \Omega\tau_0 < 1 \\ 1 & : \Omega\tau_0 = 0, \end{cases} \tag{60}$$

where

$$\begin{aligned} \Omega' &= \sqrt{\Omega^2 - \frac{1}{\tau_0^2}} \\ \frac{1}{\tau'} &= \frac{1}{\tau_0} - \sqrt{\frac{1}{\tau_0^2} - \Omega^2} \\ \frac{1}{\tau''} &= \frac{1}{\tau_0} + \sqrt{\frac{1}{\tau_0^2} - \Omega^2}. \end{aligned} \tag{61}$$

If there is no measurement or no fluctuational back action noise $(\tau_0 \to \infty)$, the system evolves according to the standard vacuum Rabi oscillation. However, if there are weak measurements $(\Omega\tau_0 > 1)$, the vacuum Rabi oscillation is already damped with a lifetime τ_0. Note that such a weak measurement is already a nonlinear measurement, in which the dynamic back action noise and the fluctuational back action noise cannot be separated. When there are strong measurements $(0 < \Omega\tau_0 < 1)$, the oscillatory behavior is completely suppressed and $\rho_z(t)$ decays monotonically toward the steady state value $\rho_z(\infty) = 1/2$. Finally, if there are exact measurements $(\Omega\tau_0 = 0)$, the system is frozen in its initial state $(\rho_z(0) = 1)$. This is the quantum Zeno effect [4] and one of the striking signature of quantum systems.

Finally, the above theoretical predictions are confirmed by the numerical simulation based on the stochastic Schrödinger wavefunction method (quantum Monte-Carlo simulation) [3]. Figure 8 shows the ensemble averaged value of finding $|\downarrow\rangle$ state (or photon number) for (a) weak measurements and (b) strong measurements. The ensemble averaged photon number vs. evolution time agrees well with Eq.(60). However, if we plot a single trajectory of the measurement result, it is a completely random scatter with zero mean for weak measurements as shown in Fig. 8(c), while it is a random telegraphic signal (or quantum jump) for strong measurements as shown in Fig. 8(d).

4.5.2 Measurement-feedback QNN

Let us consider again an optical homodyne detector in the measurement feedback QNN shown in Fig. 2, where the pre-measurement state of the internal DOPO pulse is a squeezed vacuum state with enhanced quantum noise for $\hat{x}$ and reduced quantum noise for $\hat{p}$. The out-coupled field to be detected by the homodyne detector and the transmitted field remaining in the DOPO cavity are quantum mechanically correlated in spite of the incident vacuum fluctuation from the open port of the coupler. Therefore, the post-measurement state of the transmitted field changes by reading out the measurement result for the out-coupled field. This experimental situation provides another example of approximate measurements and partial wavepacket reduction, which is schematically shown in Fig. 9(a) and (b). Note that the center position is shifted toward the measured value and the variance is reduced in the post-measurement state. If this measurement result is used to produce a feedback signal, which is injected back to the second DOPO pulse via anti-ferromagnetic coupling $(J_{ij} < 0)$ in this case, the center position of the second DOPO state is shifted to the opposite

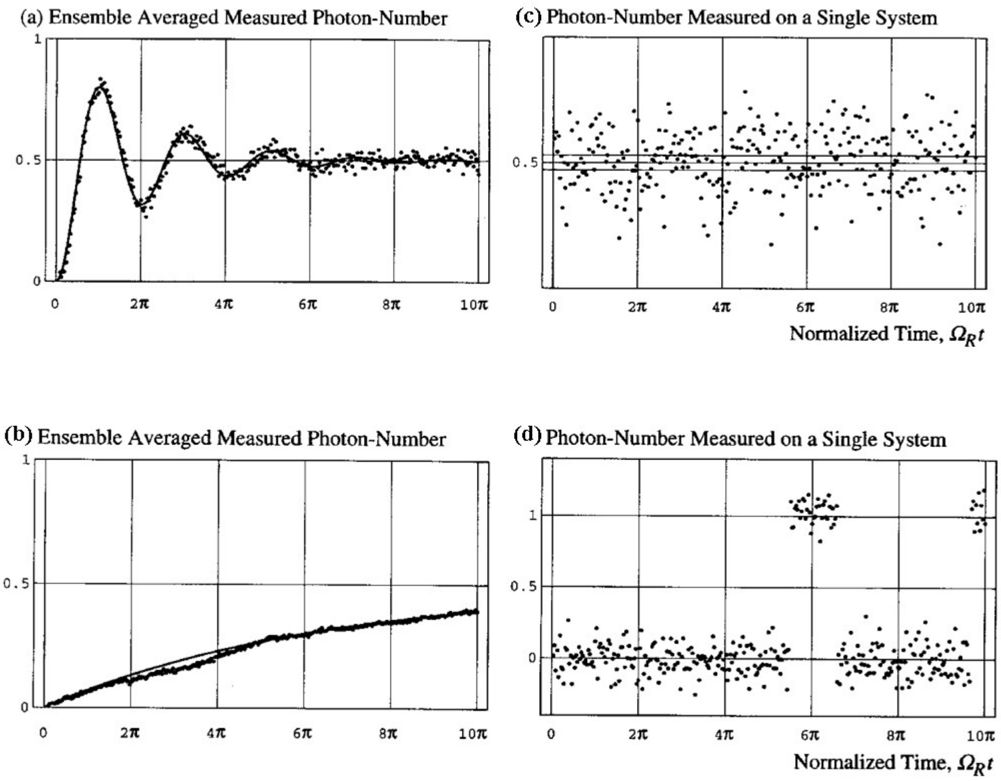

FIG. 8: The numerical simulation results for the continuously monitored vacuum Rabi oscillation. (a)(b) Ensemble averaged trajectories for weak and strong measurements. (c)(d) A single trajectory for weak and strong measurements [3].

direction from the center position of the first DOPO state, as shown in Fig. 9(b). A single measurement-feedback process already implements the anti-ferromagnetic order between the two DOPO pulses as far as the center positions of the wavefunctions are concerned.

For this case of anti-ferromagnetically coupled two DOPO pulses, we can still define the two regimes: linear continuous measurement and nonlinear continuous measurement. If the measurement strength is extremely weak (or the measurement error is very large) and the DOPO pump rate is gradually increased from below to above the threshold, the two DOPO pulses would select one of the two oscillation state ($|\uparrow\rangle$ or $|\downarrow\rangle$) randomly since the implemented anti-ferromagnetic coupling (system's ordering force) is much weaker than the noise (reservoirs' fluctuating force). The final state at a pump rate well above the threshold is therefore one of the four states, $|\uparrow\rangle|\uparrow\rangle$, $|\uparrow\rangle|\downarrow\rangle$, $|\downarrow\rangle|\uparrow\rangle$ or $|\downarrow\rangle|\downarrow\rangle$, with 25% probability. This apparently undesired limit is the regime of a linear continuous measurement. On the other hand, if the measurement strength is strong enough, the negative correlation is established in the two center positions $\langle x_1\rangle$ and $\langle x_2\rangle$ before the threshold is reached and the final state at a pump rate well above the threshold would be either $|\uparrow\rangle\,|\downarrow\rangle$ or $|\downarrow\rangle\,|\uparrow\rangle$, which are both the ground state of the Hamiltonian. This desired limit corresponds to the regime of a nonlinear continuous measurement.

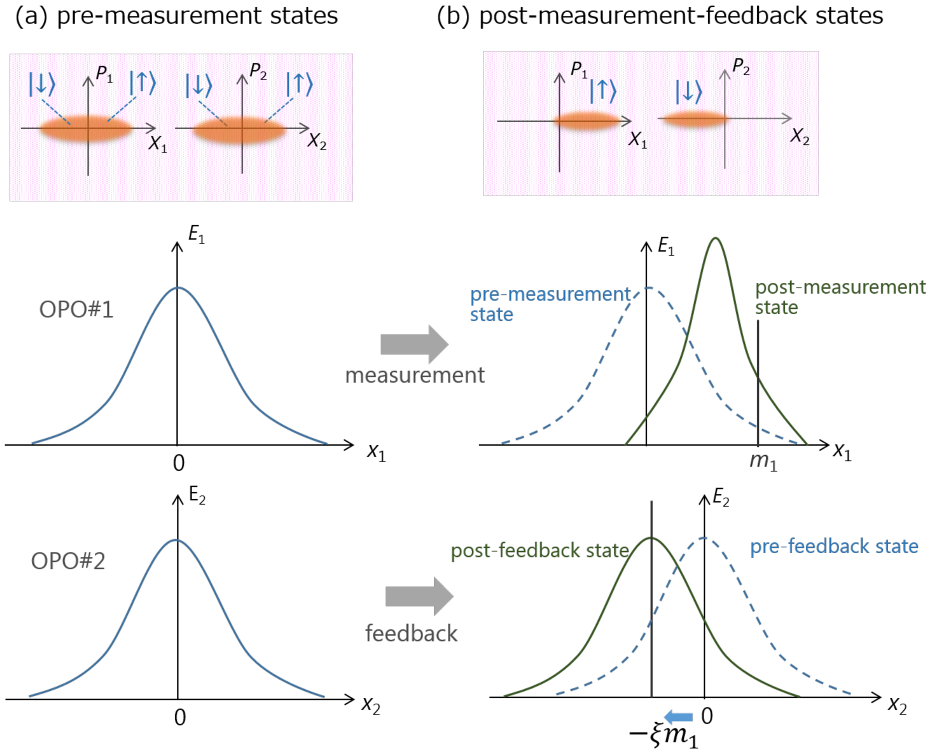

FIG. 9: A nonlinear continuous measurement in the measurement-feedback QNN with two DOPO pulses.

4.6 Contextuality in quantum measurements

It is a widely accepted quantum doctrine that an individual quantum system does not possess pre-existing values of the measured properties. Rather, a "possible" measurement result is brought into an "actual" measurement result by the joint action of a probed system and a probing apparatus. Quantum mechanics is simply silent for a single individual measurement event and only the ensemble averaged (or statistical) property of many identical measurements is a proper scientific question. This quantum doctrine appears to demand us to accept what is unmeasurable is the unreal and the nonexistent. However, not a long time ago, the people has tried to put the deeper level of description for quantum mechanics: Properties of individual systems have pre-existing values which are revealed by the measurements, even though they are hidden from us at an earlier time before the measurement. The effort to construct such an ontological interpretation of the quantum theory is known as a hidden-variable theory [5].

It was John Bell that formulated the two fundamental theorems by which the hidden variable theory disagrees with the quantum theory. In the first theorem given in 1964 [6], the failure of the hidden variable theory requires the preparation of a particular quantum state, a so-called entangled state. On the other hand, in the second theorem given in 1966 [7] and today known as the Bell-Kochen-Specker theorem [8], such preparation of the particular state is not required to show the breakdown of the hidden variable theory. In the first theorem [6], the assumption of locality plays an important role so that the non-local correlation in entangled states is indispensable for showing the discrepancy between the hidden variable theory and the quantum theory. In the second theorem [7, 8], the assumption of non-contextuality plays a similar role so that identification of a group of the observables to be measured simultaneously is crucially important.

Let us explain the concept of contextuality. The hidden variable theory tells us the measurement results for the observable $\hat{A}$ are identical for the two cases, in which $\hat{A}$ is measured simultaneously with its commuting observables $\hat{B}$, $\hat{C}$, $\cdots\cdots$ in one case or $\hat{A}$ is measured simultaneously with its other commuting observables $\hat{L}$, $\hat{M}$, $\cdots\cdots$ in another case. The quantum theory, however, predicts the two measurement results are different, so that the assumption of the pre-existing values of the measured property is denied, irrespective of the state prepared before the measurement and without the need to investigate the statistical properties of the measurement results.

The choice of either contextuality in quantum measurements or non-contextuality in classical measurements can be naively understood by the following analogy. When a politician of hesitation habit is asked if he/she supports the construction of a new nuclear power plant in his/her electoral districts, the answer is dependent on the immediate-previously asked questions. If he/she was asked about the serious nuclear accidents in the past and the cost for safety measures, the answer might be biased toward a negative one. If he/she was asked about the balance of the nation's energy resource distribution and the global warming issue, however, the answer might be biased toward a positive one. This is called "contextuality" in philosophy and a unique feature of quantum mechanics which is absent in classical counterparts.

As a concrete example for contextuality in quantum measurements, let us consider the two independent spin $-1/2$ particles (four dimensional system). The observables represented by the Pauli operators satisfy the following commutation and anti-commutation relations:

$$[\hat{\sigma}_i, \hat{\sigma}_j] = \hat{\sigma}_i\hat{\sigma}_j - \hat{\sigma}_j\hat{\sigma}_i = 2i\sum_{k=1}^{3} \varepsilon_{ijk}\hat{\sigma}_k,\ (i,\ j,\ k = x,\ y,\ z) \tag{62}$$

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}\hat{I}. \tag{63}$$

We consider the nine observables shown in Fig. 10. Then, we immediately note that it is impossible to assign the definite pre-existing values to those observables by the following argument:

1. The three observables in each row and in each column commute so that they can be measured simultaneously. This is obviously true for the top two rows and the first two columns from left. As for the bottom row and the third column from left, we can use the anti-commutation relation (Eq.(63)) to confirm it is also true for them.

2. The product of the three observables in the third column from left is -1 (based on Eq.(63)), while the product of the three observables in the other two columns and all of three rows is $+1$.

3. However, 2 is impossible to satisfy in the hidden variable theory, since the three-time row measurement results require the product of all nine values to be $+1$, while the three-time column measurement results require it to be -1.

This is the simplest version of the Bell-Kochen-Specker theorem [9]. Note that we do not need any specific state preparation for the two spin $-1/2$ particles.

Next, let us consider the set of ten observables for three independent spin $-1/2$ particles (eight dimensional system), shown in Fig. 11. We also immediately note that it is impossible to assign the definite pre-existing values to those observables by the following argument:

1. The four observables on each of the five lines of the star commute, so that they can be measured simultaneously. This conclusion is trivial except for the horizontal line, for which we can use the anti-commutation relation (Eq.(63)) twice to confirm the six pairs of the observables indeed commute.

$\hat{\sigma}_x^{(1)}$	$\hat{\sigma}_x^{(2)}$	$\hat{\sigma}_x^{(1)}\ \hat{\sigma}_x^{(2)}$
$\hat{\sigma}_y^{(2)}$	$\hat{\sigma}_y^{(1)}$	$\hat{\sigma}_y^{(1)}\ \hat{\sigma}_y^{(2)}$
$\hat{\sigma}_x^{(1)}\ \hat{\sigma}_y^{(2)}$	$\hat{\sigma}_y^{(1)}\ \hat{\sigma}_x^{(2)}$	$\hat{\sigma}_z^{(1)}\ \hat{\sigma}_z^{(2)}$

FIG. 10: A set of nine observables used to prove the contextuality in quantum measurements [9].

2. The product of the four observables on each of the four lines, except for the horizontal line, is $+1$, while the product of the four observables on the horizontal line is -1.

3. However, 2 is impossible to satisfy in the hidden variable theory, since the five line measurements will report the product of the values on each line to be -1 but each measurement result should appear twice in the product over all five lines and thus the product must be $+1$.

As is evident from the above argument, the Bell-Kochen-Specker theorem [7, 8] does not need the special properties of a particular state. The theorem can be applied to arbitrary mixed states which can exist in a highly dissipative quantum system. This is in sharp contrast to the Bell theorem [6], which requires the preparation of a high-fidelity non-local correlated state (entangled state), which can survive only in a closed (or decoherence-free) quantum system. In conclusion, the Bell-Kochen-Specker theorem rules out the assignment of non-contextual values to arbitrary observables.

The previous two thought experiments disclose a unique feature of quantum measurements. For instance, in Fig. 10, if the measurement results for the first and second observables from left in the third row are same, i.e. both $+1$ (or both -1), then the measurement result for the third observable from left in the third row should be always -1. On the other hand, if the measurement results for the first and second observables are opposite, i.e. either $(+1, -1)$ or $(-1, +1)$, then the measurement result for the third observable should be always $+1$. From this observation, we can remark that in quantum measurements, the present measurement result (for the third observable) depends on the results of the past

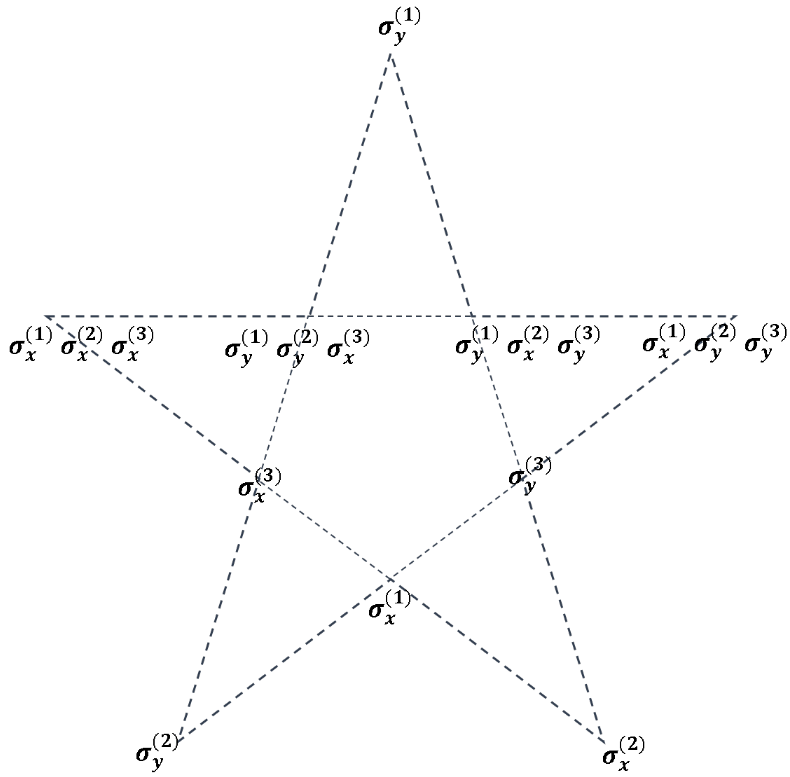

FIG. 11: A set of ten observables used to prove the contextuality in quantum measurements [9].

measurements (for the first and second observables). Let us next consider the order of measurements is switched in Fig. 10. We first measure the third observable from left in the third row, subsequently measure the second observable and finally measure the first observable. In this case, the above conclusions are still valid. Therefore, we can remark that the present measurement result (for the third observable) depends on the results of the future measurements (for the second and first observables). A quantum measurement is contextual just like a politician's mind is. We will see in the next Chapter that a similar behavior is disclosed in the measurement feedback quantum neural network, where the present-time DOPO state, either $|\uparrow\rangle$ or $|\downarrow\rangle$, depends on the measurement results in the past or in the future.

4.7 Summary

Some of the important conclusions of Chapter IV are summarized below.

1. An exact quantum measurement is fully described by the von Neumann's recipe, Eqs.(1) - (3).

2. An approximate quantum measurement with a finite error is fully described by the indirect measurement model, Eqs.(21) and (22).

3. Evolution of a quantum system under continuous measurements is governed by Eq.(44), while that under non-referred measurements (or simple dissipation) is described by Eq.(51).

4. Fluctuational back action noise and dynamic back action noise cannot be separated, in principle, in continuous quantum measurements. Quantum Zeno effort and MF-QNN are two examples for such nonlinear continuous measurements.

5. Quantum measurements are contextual: the present measurement result depends on the past measurement results and it also depends on the future measurements results.

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[1] V. B. Braginsky and F. L. Y. Khalili, Quantum Measurement (Cambridge University Press, Cambridge, 1992).

[2] W.Zurek, Phys. Rev. D24,1516 (1981)

[3] O. Alter and Y. Yamamoto, Quantum Measurement of a Single System (Wiley, New York, 2001).

[4] B. Misra ad E. C. G. Sudarshan, J. Math. Phys. 18, 756 (1977).

[5] D. Bohm, Phys. Rev. 85, 166 (1952)

[6] J. S. Bell, Physics 1, 195 (1964).

[7] J. S. Bell, Rev. Mod. Phys. 38, 447 (1966).

[8] S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).

[9] N. D. Mermin, Rev. Mod. Phys. 65, 803 (1993).

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