Stress Tensors in Surface Parameterizations

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Most of the surface parameterization methods are understood as purely geometric. However we sometimes see continuum mechanics related aspects in them. In order to indicate this, stress tensors implicitly chosen in several parameterization methods will be clarified. As a result, it is demostrated that we can switch between the methods by just altering material definitions.

1. Introduction

While most of the parameterization methods are thought to be purely geometric, there often exists a clear relation with theory of elasticity. This is very obvious because most methods first define an energy functional, which is an objective measure of distortion between reference and flattened configurations, and try to minimize it. By introducing a viewpoint of continuum mechanics, the common and different parts between the methods are clearly separated. The common part is the form of first variations of the energy functionals and the different part is the constitutive laws that gives stress tensors, i.e. material definitions.

We will analyze in this work five parameterization methods, namely discrete harmonic mapping (DHM) [9, 4], discrete conformal mapping (DCM) [7, 3], spectral conformal parameterization (SCP) [8, 1], L_2 geometric-stretch norm (L2G) [10] and a strain energy minimization with controllable distortion (CLM) [2]. Consequently, explicit expressions of stress tensors are deduced. This reinterpretation allows us to use not only triangle faces but also quadrilateral faces in surface flattening. Furthermore, we are allowed to switch between the methods by just altering material definitions. In this work we adopt Einstein summation convention.

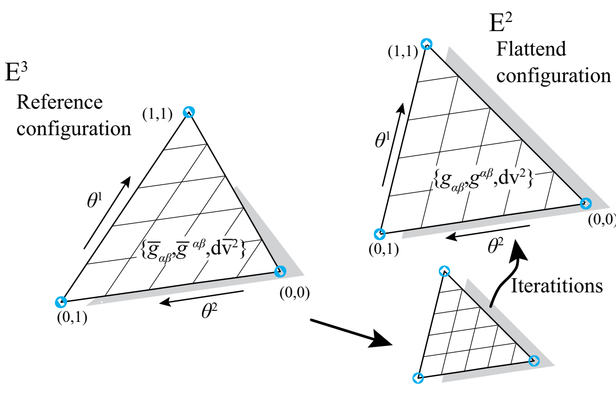

Figure 2.1. An example setup for a triangle element

2. General Formulae

We only discuss finite elements that have embedded local coordinate systems. Let $\Omega_j \subset \mathbb{R}^2$ be a domain of the j-th element. Let $(\theta^1, \theta^2) \in \Omega_j$ be a coordinate. Because (θ^1, θ^2) system is embedded, Ω_j never changes as the element deforms. For a surface mesh, we consider two configurations, namely a reference and flattened configurations. For each coordinate $(\theta^1, \theta^2) \in \Omega_j$, we assume that four 2×2 matrices $\bar{g}_{\alpha\beta}, g_{\alpha\beta}, \bar{g}^{\alpha\beta}, g^{\alpha\beta}$ are attached. The first two are the Riemannian metrics computed on the reference and flattened configurations respectively. The last two are the inverse matrices of the former two. We denote area elements by $\mathrm{d}\bar{\mathrm{v}}^2 = \sqrt{\det \bar{g}_{\mu\nu}} d\theta^1 d\theta^2$ and $\mathrm{dv}^2 = \sqrt{\det g_{\mu\nu}} d\theta^1 d\theta^2$. The quantities expressed with $\bar{\cdot}$ are constants that does not subject variational operator. Fig. 2.1 helps reader's understanding of this general setup.

On an element, a strain energy is generally expressed as

$$E_j = \int_{\Omega_j} W \left(g_{\alpha\beta}, g^{\alpha\beta}, \bar{g}_{\alpha\beta}, \bar{g}^{\alpha\beta} \right) \mathrm{d}\bar{\mathrm{v}}^2.$$

Noting that only $g_{\alpha\beta}$ is independent in W, we get

$$\delta E_j = \int_{\Omega_j} \frac{\partial W}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} \mathrm{d}\bar{\mathrm{v}}^2,$$

where δ is a variational operator. Here δ can omit $\int_{\Omega_j} d\theta^1 d\theta^2$ because we chose an embedded local coordinate system. Defining $S^{\alpha\beta} = 2\frac{\partial W}{\partial g_{\alpha\beta}}$, we can write

$$\delta E = \frac{1}{2} \int_{\Omega} S^{\alpha\beta} \delta g_{\alpha\beta} \mathrm{d}\bar{\mathrm{v}}^2.$$

This represents a small (virtual) work done by a stress filed acting on the element. In the theory of elasticity, $S^{\alpha\beta}$ is called the 2nd Piola Kirchhoff (2nd-PK in the following) stress tensor. In this work, we mainly focus on 2nd-PK stress tensor. Generally, an explicit expression of $S^{\alpha\beta}$ expressed in terms of four Riemannian metrics, i.e.

$$S^{\alpha\beta} = S^{\alpha\beta} \left(\bar{g}_{\alpha\beta}, g_{\alpha\beta}, \bar{g}^{\alpha\beta}, g^{\alpha\beta} \right),$$

is called a constitutive law. This gives a concrete material definition. Different strain energy gives a different constitutive law but the form of δE remains common. Therefore, explicitly expressing such material definitions would be beneficial for implementing a general surface parameterization framework, in which a user would switch between the methods by just altering the material definitions.

Further discretization schemes and numerical solves are not discussed in this short-paper but a standard finite element approaches can be applied.

3. Energy and stress tensors

3.1. **Discrete harmonic mapping.** For surface parameterization, a discrete harmonic mapping (DHM in the following), or so-called the cotan formula, was first introduced by Eck *et al.* [4]. Before them, the same formula was derived and applied to minimal surface problem by Pinkall *et al.* [9]. Both Eck *et al.* and Pinkall *et al.* derived DHM from a Dirichlet energy. Our expression of the Dirichlet energy is

$$E_j = \frac{1}{2} \int_{\Omega_j} g_{\alpha\beta} \bar{g}^{\alpha\beta} \mathrm{d}\bar{\mathrm{v}}^2.$$

Actually, this is very close to the expression of the Dirichlet energy in [5], to which Eck *et al.* referred. The corresponding 2nd-PK stress tensor is

$$S^{\alpha\beta} = \bar{g}^{\alpha\beta}. \tag{3.1}$$

This material definition means that, the element is always tensioned and hence all the boundary vertices must be fixed.

3.2. **Discrete conformal mapping.** Levy *et al.* and Desbrun *et al.* independently derived [7, 3] discrete conformal mappings (DCM in the following). Later, Floater *et al.* and Sheffer *et al.* pointed out [6, 12] the equivalence of these two.

In [6], DCM is characterized by a conformal energy defined by

$$E_C = E_D - S,$$

where E_D is the Dirichlet energy and S is the surface area. It has been known that $E_D \geq S$ and the equality holds when the flattened surface is conformal to the reference surface. If a surface is approximated by discrete elements, $E_C = 0$ is hardly achieved. Hence, instead of $E_C = 0$, $E_C \to \min$ is solved in DCM.

Because the area of an element is expressed as

$$S_j = \int_{\Omega_j} \mathrm{dv}^2 = \int_{\Omega_j} \mathrm{J}\mathrm{d}\bar{\mathrm{v}}^2, \text{ where } \mathrm{J} = \frac{\sqrt{\det g_{\mu\nu}}}{\sqrt{\det \bar{g}_{\mu\nu}}},$$

our expression for E_C is

$$E_j = \int_{\Omega_j} \left(\frac{1}{2} \bar{g}^{\alpha\beta} g_{\alpha\beta} - \mathrm{J} \right) \mathrm{d}\bar{\mathrm{v}}^2.$$

Using $\delta \mathrm{J} = \frac{1}{2} g^{ij} \delta g_{ij} \mathrm{J}$, we get the corresponding 2nd-PK stress tensor as

$$S^{\alpha\beta} = \left(\bar{g}^{\alpha\beta} - g^{\alpha\beta} \mathrm{J} \right). \tag{3.2}$$

Both Levy *et al.* and Desbrun *et al.* noted that at least two points should be fixed in DCM. Levy claimed that the ideal number of fixed vertices is 2. However, if two vertices are fixed, the conformality would be distorted because of the reaction forces acting on the fixed vertices.

3.3. **Spectral conformal parameterization.** Because DCM requires at least two vertices fixed, a method that does not require fixed vertices is expected. Mullen *et al.* proposed [8] a spectral conformal parameterization (SCP in the following), which is an extension of DCM, and which does not require fixed vertices. Later Alexa *et al.* polished [1] the method and proposed constraining the total area of a surface mesh instead of fixing two vertices. This approach is characterized by

$$\begin{aligned} E_C(\boldsymbol{x}) = E_D(\boldsymbol{x}) - S(\boldsymbol{x}) \to \min, \\ \text{s.t. } S(\boldsymbol{x}) = \bar{S}, \end{aligned}$$

where $\bar{S}$ is a prescribed total area to which S is constrained. Alexa *et al.* reduced this problem to a generalized eigen-value problem as well as Mullen *et al.* Instead, we preferred to deduce a stress tensor from this problem. Instead of applying the Lagrange multiplier method, we used a penalty term and our reinterpretation is

$$E_j = \sum_j \int_{\Omega_j} \left(\frac{1}{2} \bar{g}^{\alpha\beta} g_{\alpha\beta} - \mathrm{J} \right) \mathrm{d}\bar{\mathrm{v}}^2 + \frac{1}{2\bar{S}} \left(S - \bar{S} \right)^2 \to \min,$$

Because

$$\delta E = \sum_j \frac{1}{2} \int_{\Omega_j} \left(\bar{g}^{\alpha\beta} + \left(\frac{S}{\bar{\bar{S}}} - 2 \right) g^{\alpha\beta} \mathrm{J} \right) \delta g_{\alpha\beta} \mathrm{d}\bar{\mathrm{v}}^2$$

we get the corresponding 2nd-PK stress as

$$S^{\alpha\beta} = \bar{g}^{\alpha\beta} + \left(\frac{S}{\bar{\bar{S}}} - 2 \right) g^{\alpha\beta} \mathrm{J}. \tag{3.3}$$

Note that S must be computed in advance of the computation of $S^{\alpha\beta}$.

3.4. L_2 **Geometric stretch norm.** Sander *et al.* have been using [11, 10] an L_2 geometric stretch norm for surface parameterization (L2G in the following). Our expression for L2G is

$$E_j = \frac{1}{2} \int_{\Omega_j} \bar{g}_{\alpha\beta} g^{\alpha\beta} \mathrm{d}\bar{\mathrm{v}}^2.$$

Using $\frac{\partial A^{-1}}{\partial x} = -A^{-1} \frac{\partial A}{\partial x} A^{-1}$, where A is a non-singular matrix, we get the corresponding 2nd-PK stress as

$$S^{\alpha\beta} = -\bar{g}_{\mu\nu} g^{\mu\alpha} g^{\beta\nu}. \tag{3.4}$$

The minus sign in $S^{\alpha\beta}$ indicates that elements are always in a compression state. Hence, as well as DHM, all the boundary vertices have to be fixed.

3.5. **Strain energy minimization with controllable distortion.** Clarenz *et al.* developed [2] a special material definition for parameterization (CLM in the following). Their work is very important in that it indicated the possibility of designing special materials for each particular objective. The material has many parameters $\{\alpha_l, \alpha_a, \alpha_c, \beta, p, r, s, t\}$ but the authors mainly discuss the cases with $p = r = s = 2$, $t = 1$. Additionally, it is indicated that $\beta = \alpha_l/\alpha_a + 1$. Thus the parameters are reduced to $\{\alpha_l, \alpha_a, \alpha_c\}$. These parameters are used to control the length and area preservability, and conformality, respectively. In the following, we denote $\{\alpha_l, \alpha_a, \alpha_c\}$ by $\{w_L, w_A, w_C\}$. Because Clarenz *et al.*'s work was also based on theory of elasticity, our expression for this special material is very close to the original expression and it is

$$E_j = \int_{\Omega_j} \left[w_L I_C + w_A \left(\mathrm{J}^2 + \beta \mathrm{J}^{-2} \right) + w_C \left(I_C^2 \mathrm{J}^{-2} - 4 \right) \right] \mathrm{d}\bar{\mathrm{v}}^2,$$

where $I_C = g_{ij} \bar{g}^{ij}$. By using $\delta I_C = \bar{g}^{\alpha\beta} \delta g_{\alpha\beta}$ and $\delta \mathrm{J} = \frac{1}{2} \mathrm{J} g^{\alpha\beta} \delta g_{\alpha\beta}$, we get

$$S^{\alpha\beta} = w_L S_L^{\alpha\beta} + w_A S_A^{\alpha\beta} + w_C S_C^{\alpha\beta}, \tag{3.5}$$

where

$$S_L^{\alpha\beta} = \bar{g}^{\alpha\beta} - g^{\alpha\beta} \mathrm{J}^{-2}$$

$$S_A^{\alpha\beta} = g^{\alpha\beta} \left(\mathrm{J}^2 - \mathrm{J}^{-2} \right)$$

$$S_C^{\alpha\beta} = 2 \left(\bar{g}^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} I_C \right) I_C \mathrm{J}^{-2}.$$

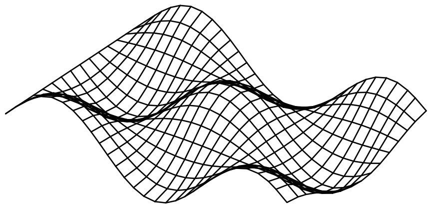

FIGURE 4.1. Example mesh surface

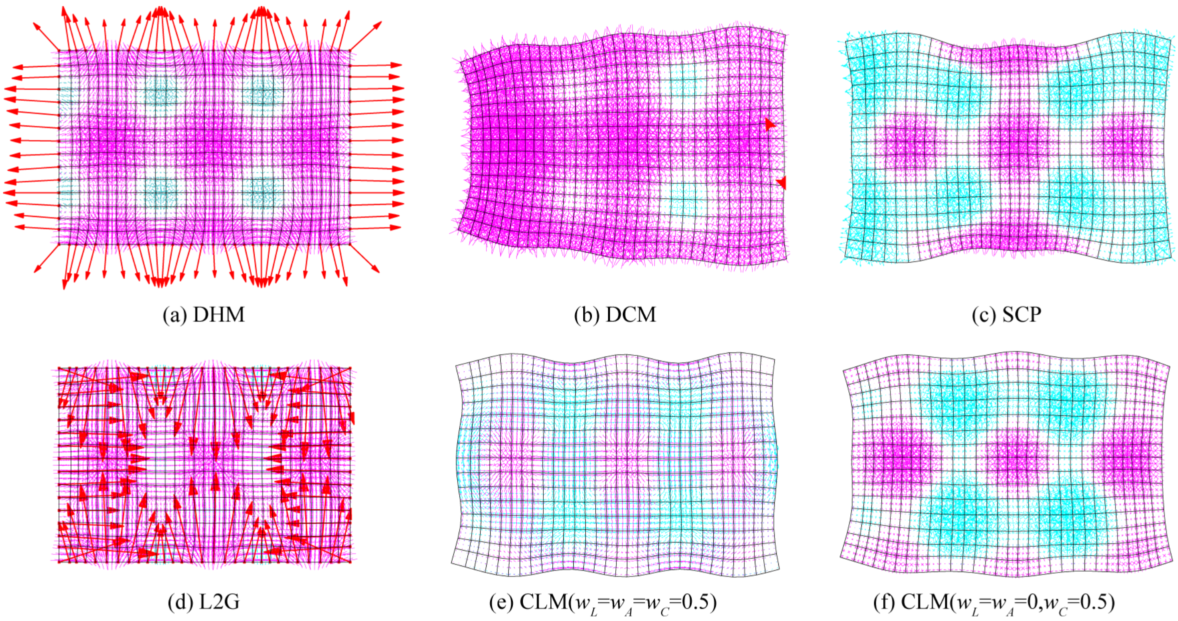

FIGURE 4.2. Results

4. EVALUATION

Aiming to compare the deduced stress tensors, we conducted direct energy minimization with a quadrilateral mesh surface shown in Fig. 4.1. For evaluation, an objective measure of distortion is needed and it should not be stress tensors. We simply used a large strain tensor defined by

$$E_{ij} = \frac{1}{2}\left(g_{ij} - \bar{g}_{ij}\right). \tag{4.1}$$

Because matrix E_{ij} itself does depend on local coordinate system, we carried out a generalized eigen-value decomposition, which is characterized by

$$\left(g_{ij} - \bar{g}_{ij}\right) V^j = \lambda g_{ij} V^j.$$

From V^j, we computed

$$\boldsymbol{V} = \frac{V^j \boldsymbol{g}_j}{\sqrt{V^\alpha V^\beta g_{\alpha\beta}}},$$

where $\boldsymbol{g}_1, \boldsymbol{g}_2$ are base vectors adjunct with (θ^1, θ^2) system in a flattened configuration. Note that $\boldsymbol{V}$ is normalized. As the result, two pairs of principal strain and directions$\{\lambda_1, \boldsymbol{V}_1\}$ and $\{\lambda_2, \boldsymbol{V}_2\}$ are obtained. Note that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are orthogonal to each other. When signs of eigen-values meet and $|\boldsymbol{V}_1| = |\boldsymbol{V}_2|$ is achieved at somewhere in a mesh surface, the flattened mesh surface is locally conformal to the reference mesh surface at that point. When $\lambda_i > 0$, this means that the element is locally stretched along $\boldsymbol{V}_i$ and if $\lambda_i < 0$, it is locally compressed. Fig. 4.2 shows minimization results of (a) DHM, (b) DCM, (c) SCP, (d) L2G, (e) CLM with $w_L = w_A = w_C = 0.5$, (f) CLM with $w_L = w_A = 0$, $w_C = 0.5$. In the figure, $\{\lambda_1 \boldsymbol{V}_1, \lambda_2 \boldsymbol{V}_2\}$ are plotted with magenta color when $\lambda_i > 0$ and with cyan color when $\lambda_i < 0$. If some vertices are fixed, reaction forces acting on the vertices are also plotted with red color.

During direct minimization, it was possible to obtain different parameterization results by switching material definitions without terminating computation.

5. Conclusion

We deduced explicit expressions of stress tensors that have not been yet clearly represented in the previous works. Now, because only material definitions are uncommon parts between the methods and other parts remain common. Consequently, we are allowed to switch between the methods by just replacing material definitions.

References

[1] Marc Alexa and Max Wardetzky. Discrete Laplacians on general polygonal meshes. *ACM Transactions on Graphics*, 30(4):1, July 2011.

[2] U Clarenz, N Litke, and M Rumpf. Axioms and variational problems in surface parameterization. *Computer Aided Geometric Design*, 21(8):727–749, 2004.

[3] M Desbrun, M Meyer, and P Alliez. Intrinsic parameterizations of surface meshes. *Computer Graphics Forum*, 21(3):209–218, 2002.

[4] Matthias Eck, Tony DeRose, Tom Duchamp, Hugues Hoppe, Michael Lounsbery, and Werner Stuetzle. Multiresolution analysis of arbitrary meshes. In *Proceedings of the 22nd annual conference on Computer graphics and interactive techniques - SIGGRAPH '95*, pages 173–182, New York, USA, September 1995. ACM.

[5] J Eells and JH Sampson. Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics*, 86(1):109–160, 1964.

[6] MS Floater and K Hormann. Surface parameterization: a tutorial and survey. In Neil A. Dodgson, Michael S. Floater, and Malcolm A. Sabin, editors, *Advances in Multiresolution for Geometric Modelling*, pages 157–186. Springer Berlin Heidelberg, 2005.

[7] B Lévy, S Petitjean, N Ray, and J Maillot. Least squares conformal maps for automatic texture atlas generation. *ACM Transactions on Graphic*, 21(3):362–371, 2002.

[8] Patrick Mullen, Yiying Tong, Pierre Alliez, and Mathieu Desbrun. Spectral Conformal Parameterization. *Computer Graphics Forum*, 27(5):1487–1494, July 2008.

[9] U Pinkall and K Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental mathematics*, 2(1):15–36, 1993.

[10] Pedro V. Sander, Steven J. Gortler, John Snyder, and Hugues Hoppe. Signal-specialized parametrization. In *Proceedings of the 13th Eurographics workshop on Rendering*, EGRW '02, pages 87–98, July 2002.

[11] Pedro V. Sander, John Snyder, Steven J. Gortler, and Hugues Hoppe. Texture mapping progressive meshes. In *Proceedings of the 28th annual conference on Computer graphics and interactive techniques*, SIGGRAPH '01, pages 409–416, New York, New York, USA, August 2001. ACM Press.

[12] Alla Sheffer, Bruno Lévy, Maxim Mogilnitsky, and Alexander Bogomyakov. ABF++: fast and robust angle based flattening. *ACM Transactions on Graphics*, 24(2):311–330, April 2005.