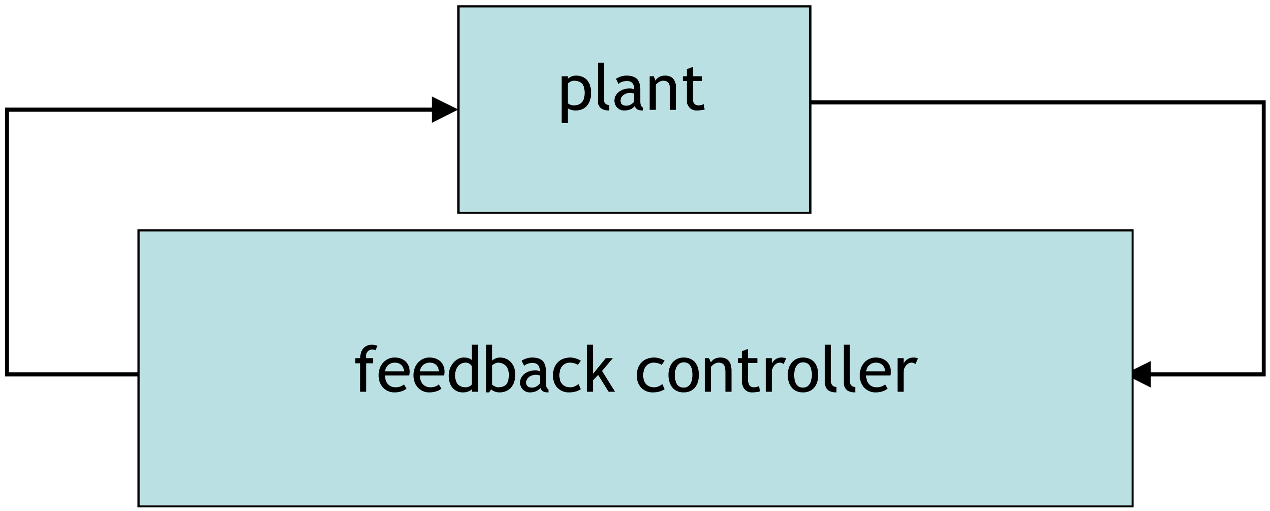
State observers in real-time feedback control



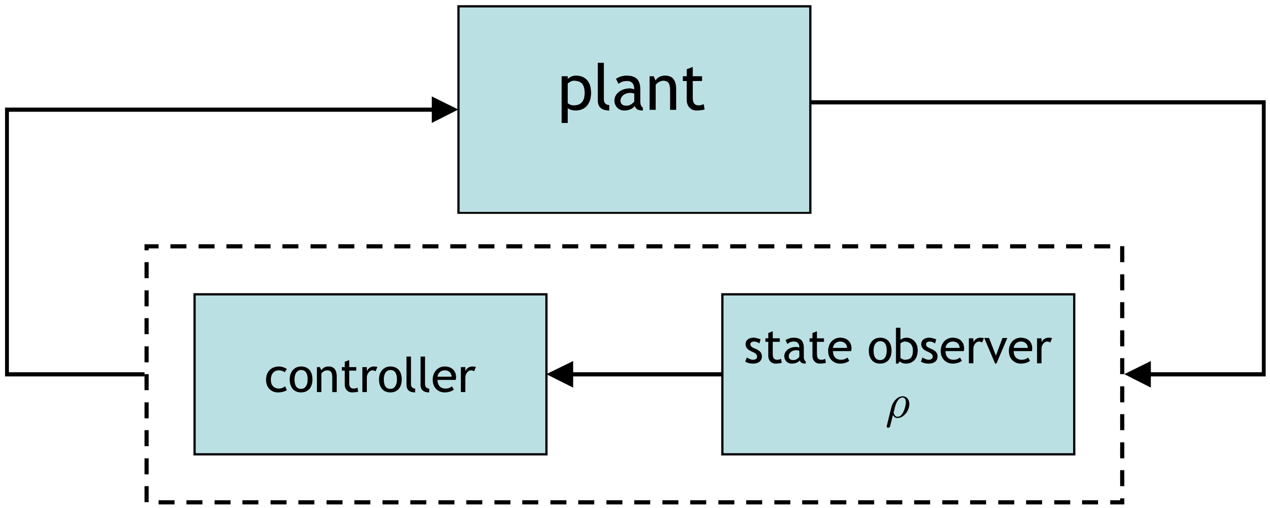
Based on:
- *Model* of plant intrinsic dynamics and exogenous disturbances,
- Knowledge of (noisy, partial) measurement record over some past time interval,

Goals:
- Predict statistics of possible future measurements as accurately* as possible,
- Utilize feedback to alter these statistics in desired ways.

Controller can maintain jointly sufficient statistics of the past measurement records, updated recursively: the *information state*. In the Kalman filter, e.g., the conditional mean and variance are sufficient statistics for a Gaussian posterior probability distribution on the plant coordinate.

*The most common – but not unique – measure of accuracy is mean-square error.

Quantum filtering: ρ as the *information state*



Quantum measurement-based feedback:

- Information state $\leftrightarrow$ ρ_c
- Recursive filter $\leftrightarrow$ Stochastic Master/Schrodinger Equation
- In the canonical design problem, the plant (S,L,H) is given and we are free to choose a specific way of monitoring the output fields (*e.g.*, homodyne or photon counting) and to design a control law with real-time actuation of plant input fields and/or adjustable parameters in the plant Hamiltonian
- The usual, unconditional Master Equation for the plant can be viewed as the "open-loop" Master Equation
- If we implement our measurement-feedback scheme and then re-average over all the noises, we should (formally speaking) obtain a "closed-loop" Master Equation with different properties (although this is not generally feasible in practice)

Quantum filtering & measurement-feedback control

L. Bouten, R. van Handel and M. R. James, SIAM Review **51**, 239 (2009); math.PR/0606118
L. Bouten and R. van Handel, math-ph/0511021

Itô amplitude-quadrature homodyne Stochastic Master Equation / Kushner-Stratonovich Equation:

$$d\rho_t = \underbrace{-i[H,\rho_t]\,dt + \sum_{j=1}^{n}\{\mathcal{D}[L_j]\rho_t\}\,dt}_{\text{“predictor” – the unconditional ME}} + \underbrace{\sqrt{\eta}\,\mathcal{H}[L_1]\rho_t\,\overline{dW}_t}_{\text{“corrector” – innovation term}}$$

- (S,L,H) model has *n* input-output channels
- We are monitoring channel 1 with quantum efficiency η

$$\overline{dW}_t = dy_t - \sqrt{\eta}\,\mathrm{Tr}[(L_1 + L_1^\dagger)\rho_t]\,dt, \qquad dy_t = I_{\mathrm{hom}}\,dt$$

$$\mathcal{D}[c]\rho \equiv c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c), \qquad \mathcal{H}[c]\rho \equiv c\rho + \rho c^\dagger - \mathrm{Tr}[(c + c^\dagger)\rho]\rho$$

For least-squares-optimal recursive filter, dW_t is Gaussian white with variance dt

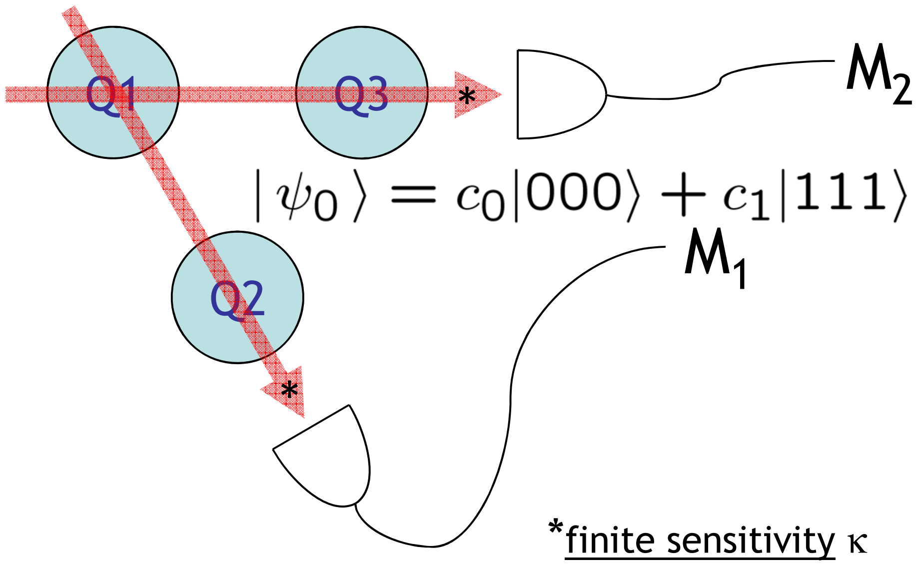
$$\text{“}\quad I_{\mathrm{hom}}\,dt \sim \sqrt{\eta}\,\mathrm{Tr}[(L_1 + L_1^\dagger)\rho_t]\,dt + dW_t \quad\text{”}$$

Coding and continuous syndrome measurement

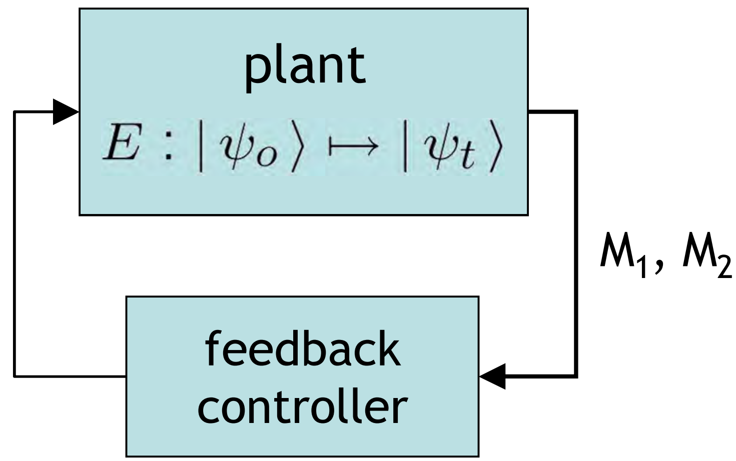
Continuous-time "relaxations" of QEC (Ahn, Doherty and Landahl, PRA 65, 042301, 2002)

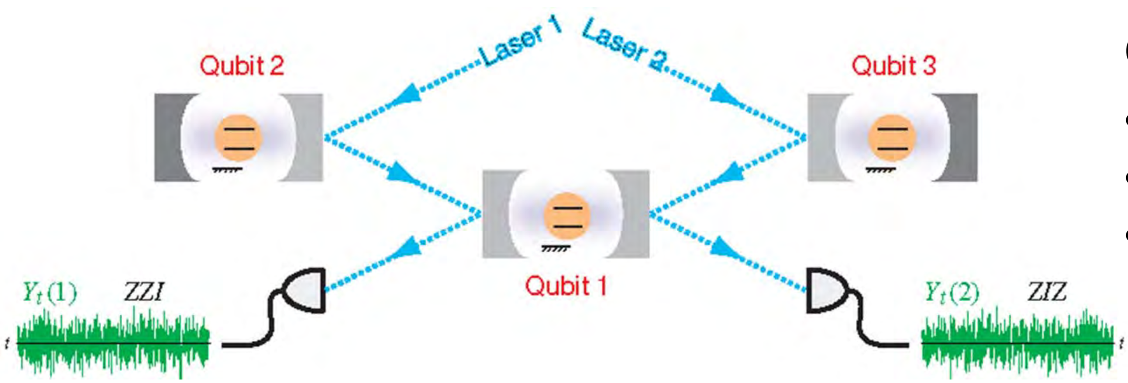
- Encode state in a stabilizer code
- Continuous QND measurement of syndrome

Pedagogical example: three-qubit bit flip code



*finite sensitivity κ

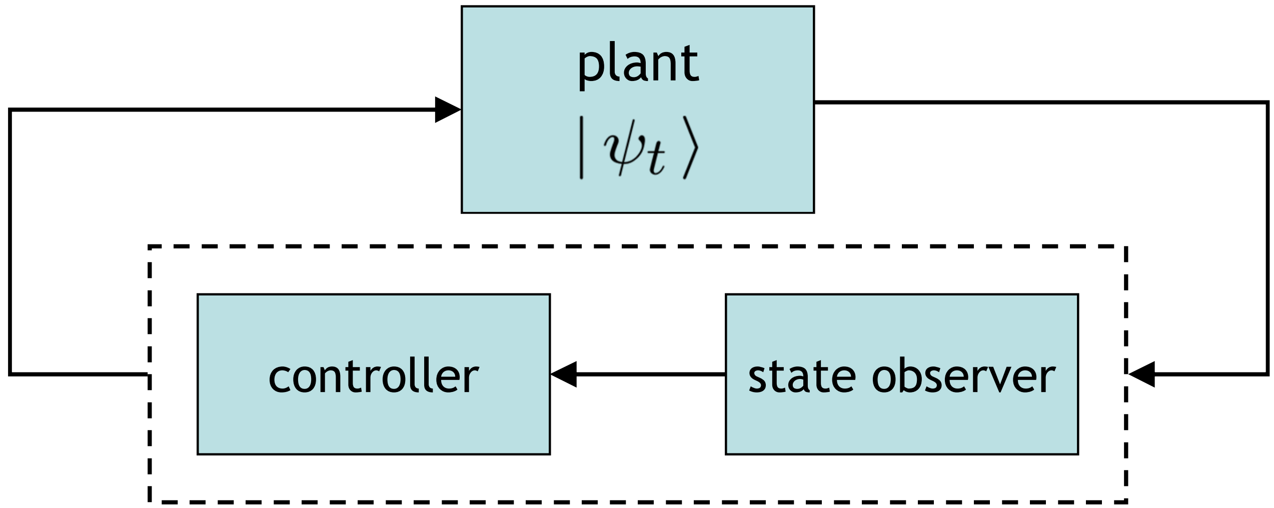




Continuous syndrome measurement:

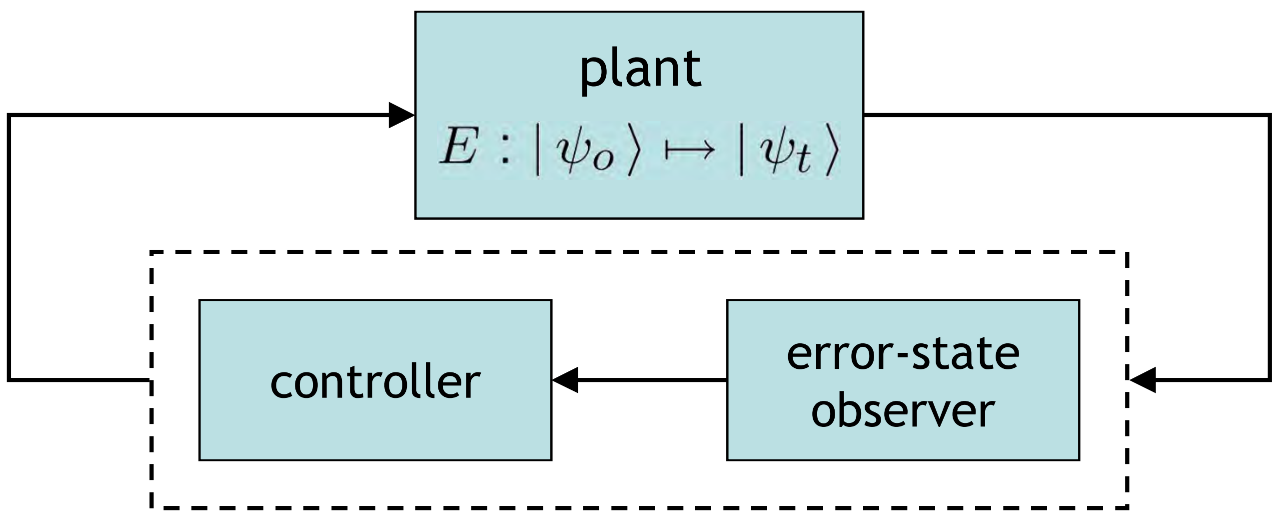
- constant qubit-cavity couplings
- cw coherent-state laser probes
- homodyne detection

State observer $\to$ error-state observer



Goal:
$|\psi_t\rangle = |\psi_o\rangle$

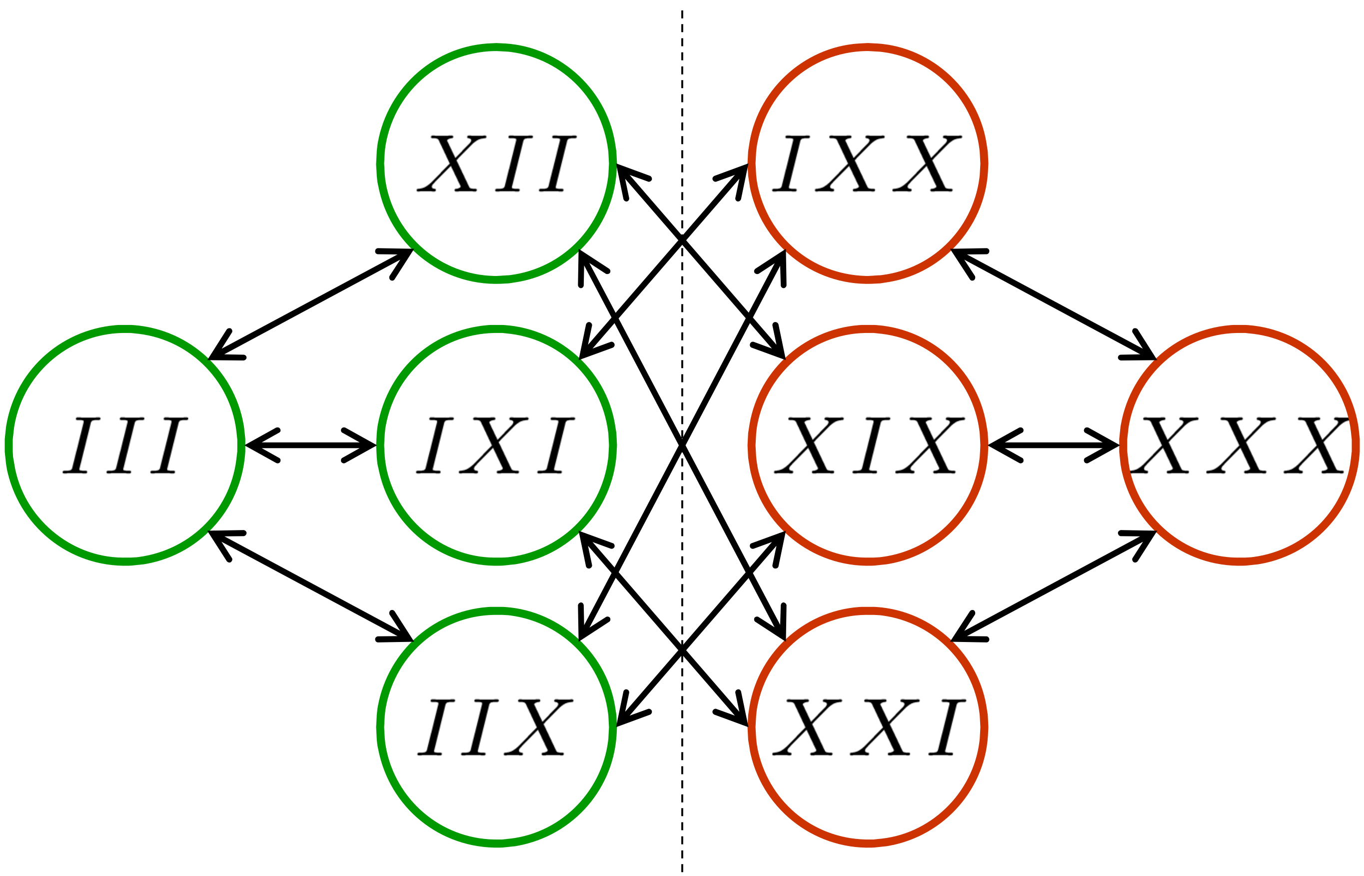
measurement should yield full information on *E* but none on encoded state



Goal:
$E = 1$

‘Error-state graph’ for the bit-flip code

$$M_1 = Z \otimes Z \otimes I \qquad M_2 = Z \otimes I \otimes Z$$

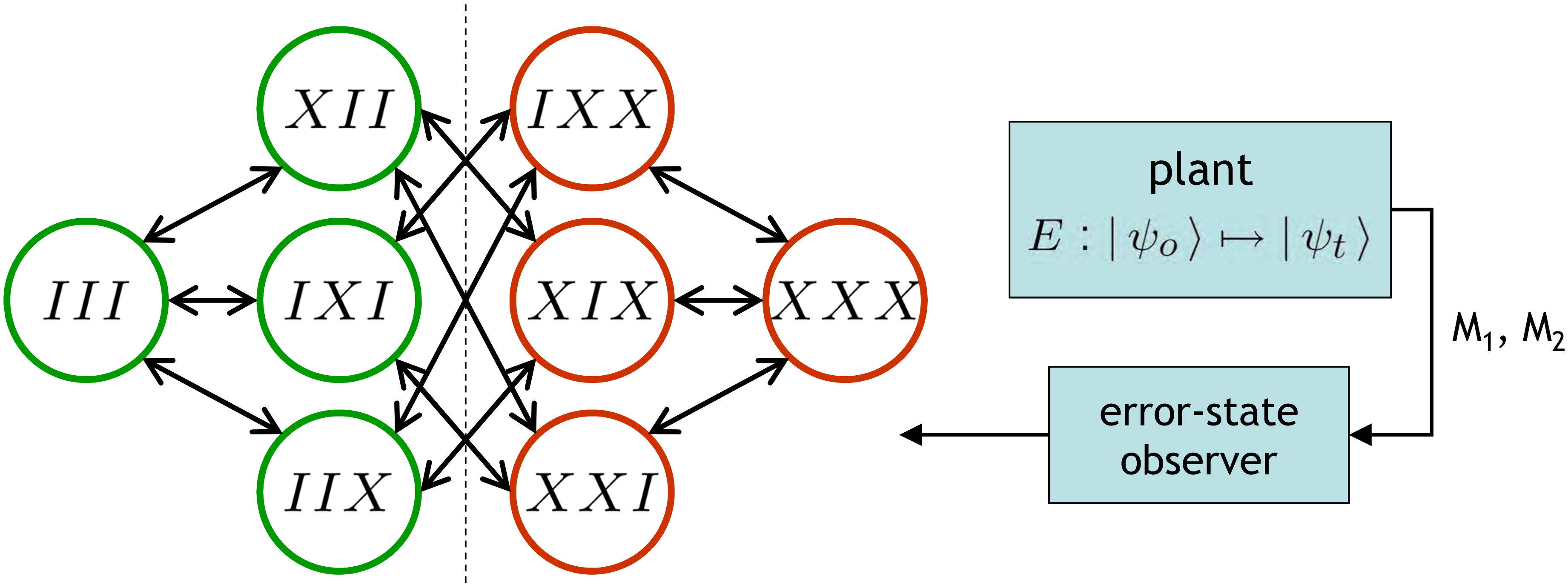


- Continuous QND syndrome measurement $\Rightarrow$ Markov jump dynamics for error state
- Mapping of error state to syndrome is degenerate

Error-state tracking with a Wonham filter

Ramon van Handel and HM, quant-ph/0511221

Assertion (numerically testable via comparison to SME): optimal filter for the error state can be derived as a *Wonham filter* (Wonham, 1965) for the induced Markov jump process of the error state

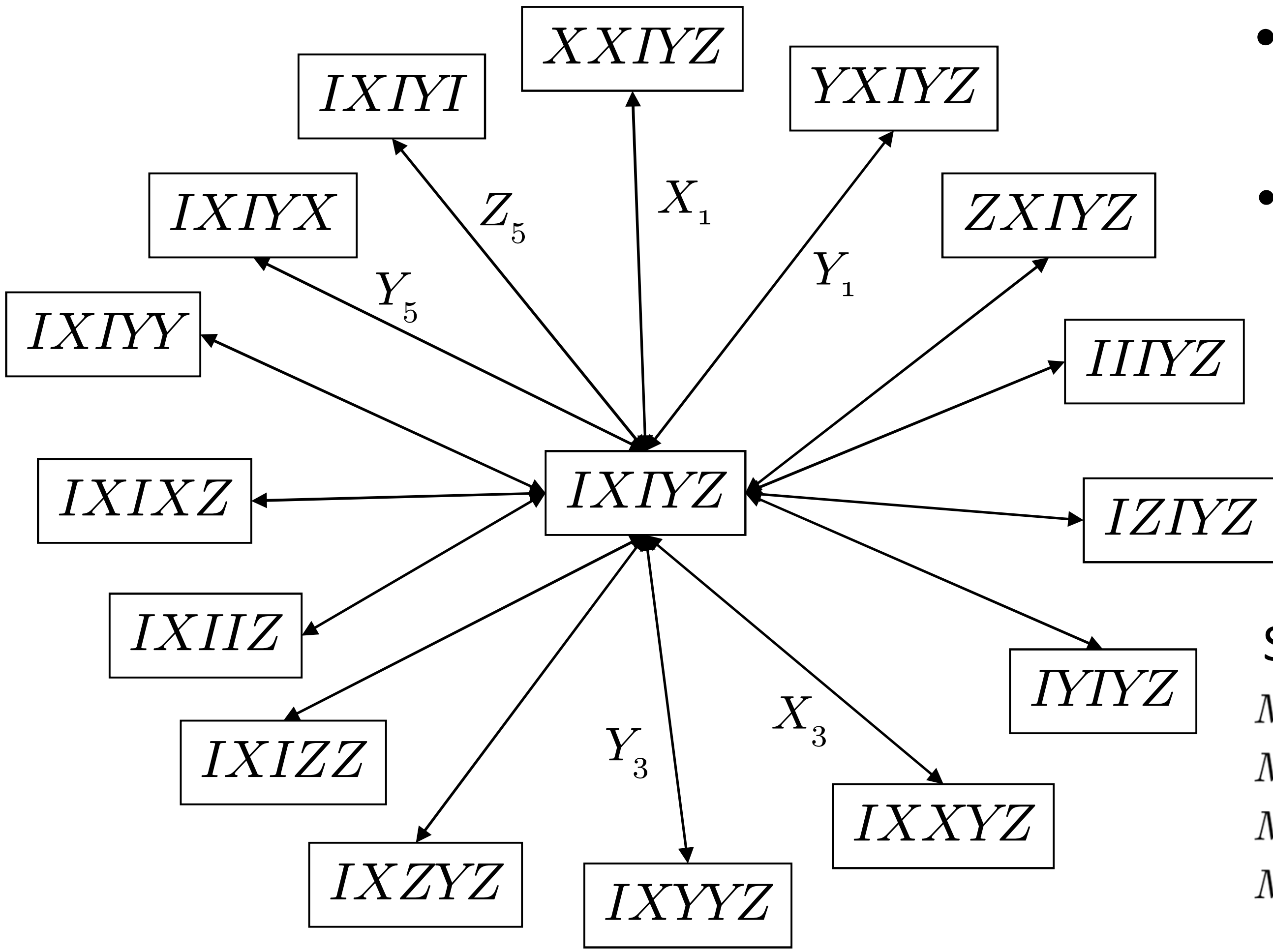


$$dp_j = \left(\textstyle\sum_{i\neq j} \nu_{ij} p_i - \nu_j p_j\right)dt + \textstyle\sum_k \beta_k^{-2} p_j (a_{jk} - \langle M_k \rangle)(M_k - \langle M_k \rangle)$$

- nonlinear filter, much studied in "hybrid stochastic" control theory

Filter *stability* results: P. Chigansky and R. van Handel, "Model robustness of finite state nonlinear filtering over the infinite time horizon," Ann. Appl. Probab. 17, 688 (2007).

Error-state graph for the five-qubit code



- 4^5=1024 error states (I,X,Y,Z for each qubit)
- 3×5=15 edges/state (X,Y,Z for each qubit)

Stabilizer generators:

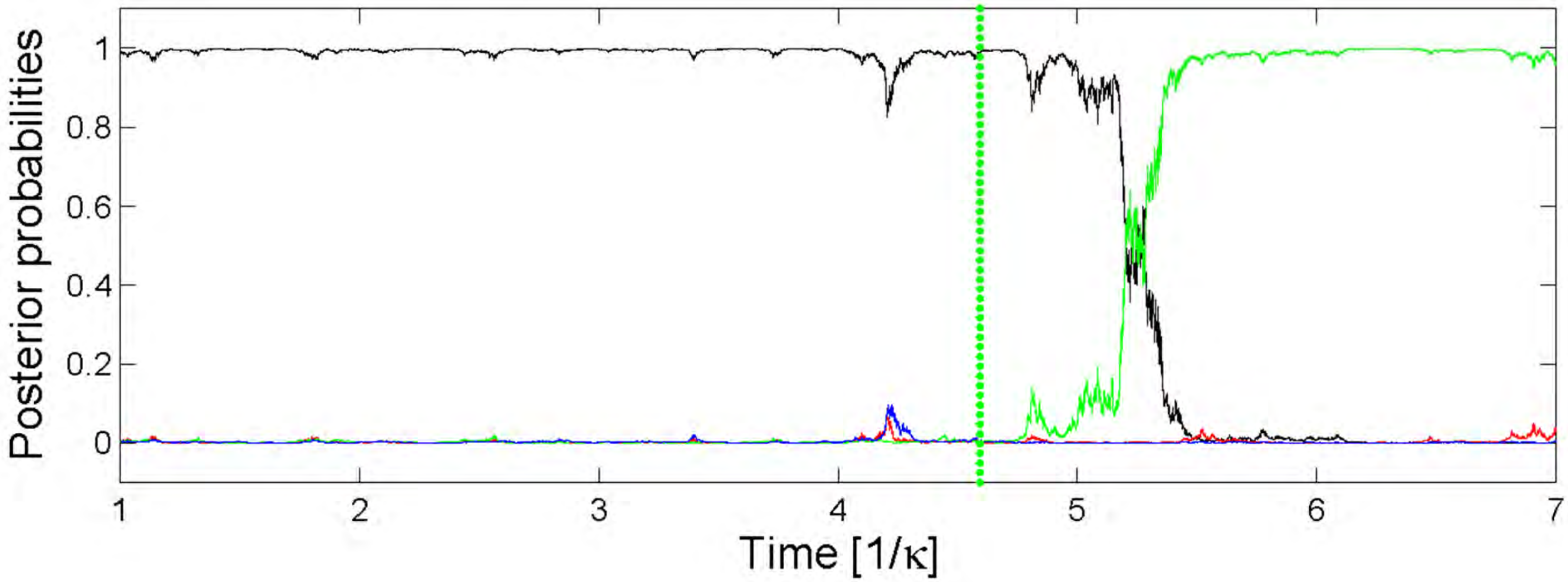
$$M_1 = X\otimes Z\otimes Z\otimes X\otimes I$$
$$M_2 = I\otimes X\otimes Z\otimes Z\otimes X$$
$$M_3 = X\otimes I\otimes X\otimes Z\otimes Z$$
$$M_4 = Z\otimes X\otimes I\otimes X\otimes Z$$

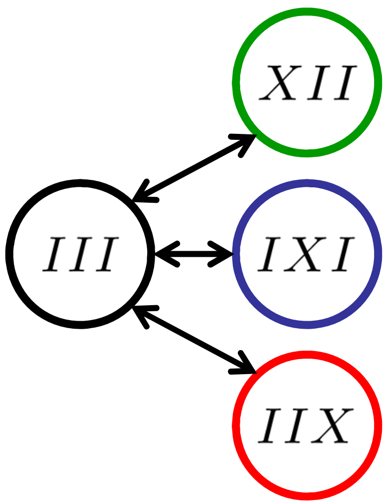
Stochastic Master Equation:

$$\kappa = 1,\ dY_j = 2\langle M_j\rangle dt + dW_j$$

$$d\rho = \sum_{\substack{i=1\\ \alpha\in\{x,y,z\}}}^{5} \gamma(\sigma_i^\alpha \rho\, \sigma_i^\alpha - \rho)dt + \sum_{j=1}^{4} \left\{(M_j\rho M_j - \rho)dt + (M_j\rho + \rho M_j - 2\langle M_j\rangle)dW_j\right\}$$

Jump dynamics of the error state





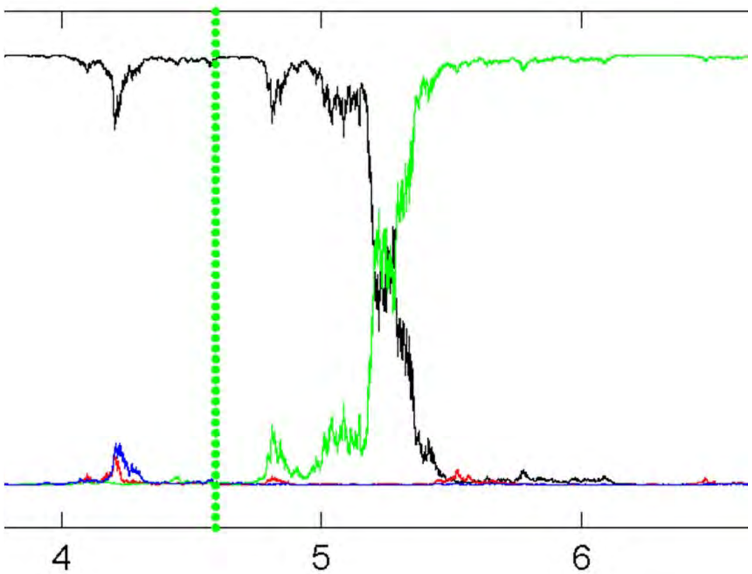
Continuous syndrome measurement localizes the error state; bit-flip decoherence induces jump-like transitions

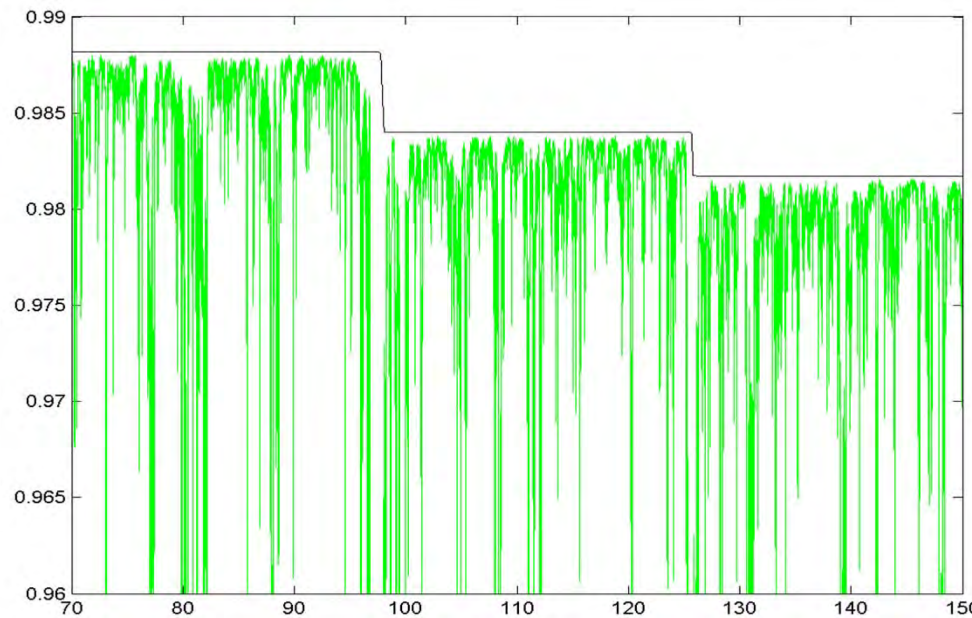
Finite measurement strength/sensitivity gives rise to detection delay and quiescent fluctuations

Purity of conditional distribution for the error state

maximize $\langle \psi_{T+\varepsilon} | \psi_0 \rangle$ "know E as well as possible" → *entropy* or *purity* of conditional distribution for the error state

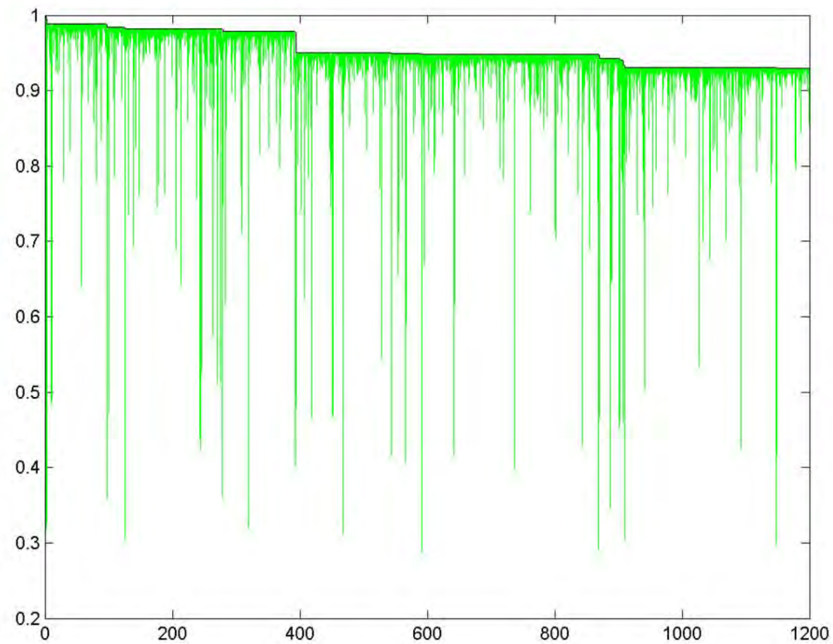
- If we know the error state with certainty we can recover perfectly
- Largest conditional probability $\max_i\{p_i\}$ directly related to decoded fidelity
- Purity lost with time because of multiple errors within detection delay
- Purity not monotonic because of transitions and quiescent fluctuations

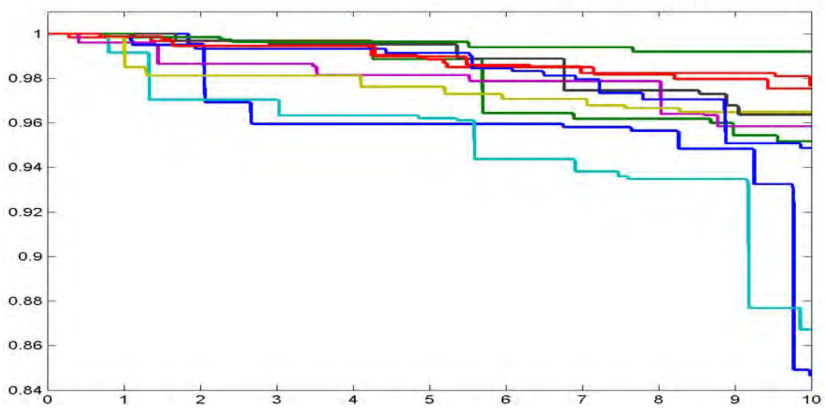


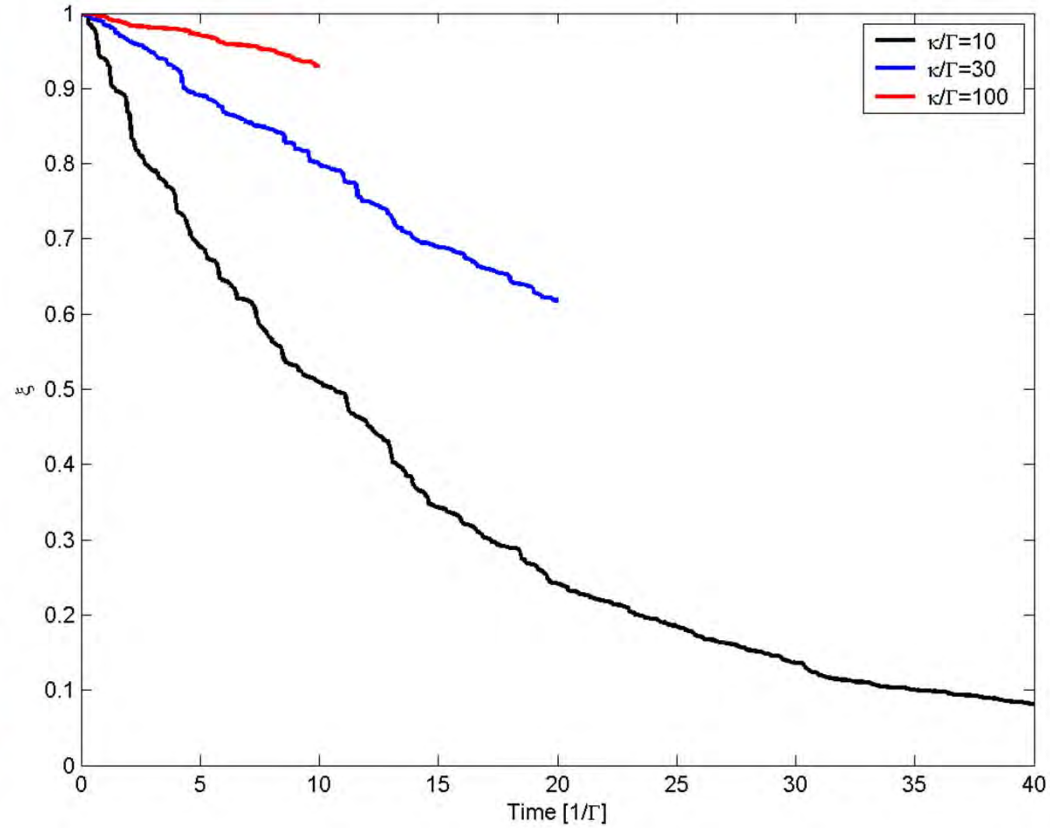


$\xi \equiv \max_i\{p_i / \Pr[\vec{a}_i]\}$ "holds" future supremum of $\max_i\{p_i\}$

Simulation results for symmetric Pauli decoherence



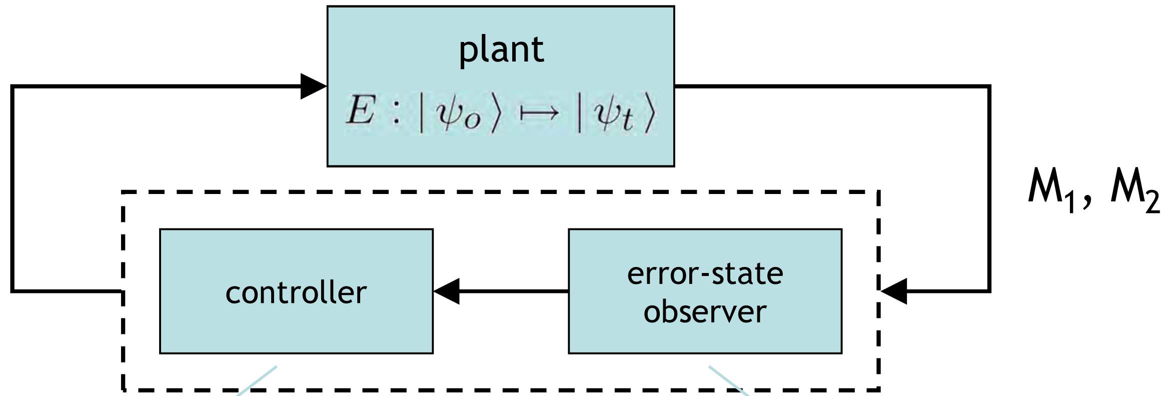


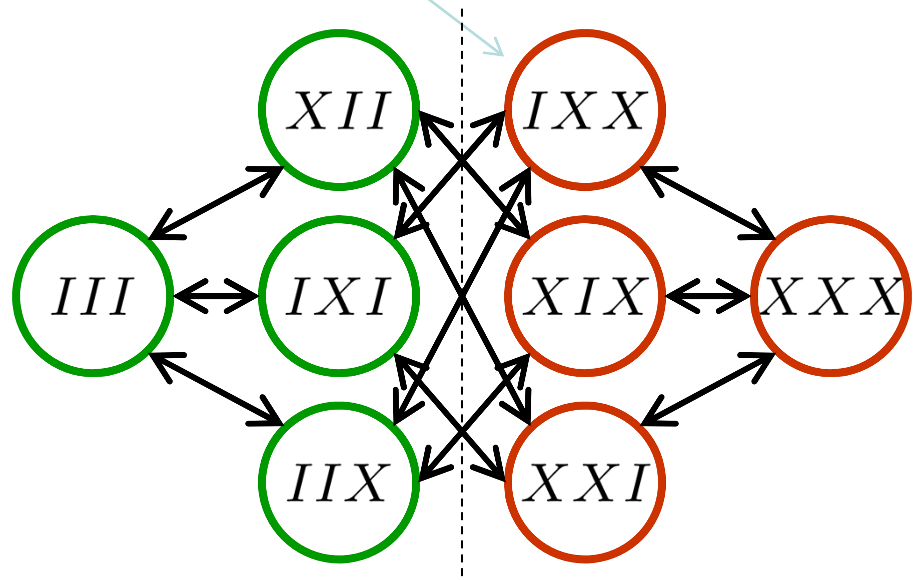


- Protocol: when data is "recalled," wait for $\max_i\{p_i\}$ to approach ξ and then decode
- Further work required to derive optimal decision policy

Quantum memory with separated control strategy

Goal:
$E = 1$





$$dp_j = (\sum_{i \neq j} \nu_{ij} p_i - \nu_j p_j)dt + \sum_k \beta_k^{-2} p_j (a_{jk} - \langle M_k \rangle)(M_k - \langle M_k \rangle)$$

Correction Strategies

An **error correction strategy** consists of the following:

1. An increasing sequence of times $\{\vartheta_n\}$ at which we correct.
2. A sequence of corrections $\{\zeta_n\}$ to perform at time n.
3. **Constraint:** ϑ_n and ζ_n may depend on the syndrome observations but only in a *causal* manner (i.e., the decision to correct at a certain time may only depend on the past observation history).

Queueing model: we do not know in advance when the memory will be accessed, so we presume that it will be read out at a *random* time τ.

The **cost** of a correction strategy balances our conflicting goals:

$$J_C[\{\vartheta_n, \zeta_n\}] = \mathbf{P}[\text{Wrong syndrome at time } \tau] + C\ \mathbf{E}[\#\{n : \vartheta_n \leq \tau\}].$$

If $C > 0$ is large, then we give more weight to minimizing the total number of corrections. When $C > 0$ is small, we give more weight to being in the correct syndrome at the readout time.

Optimal Control

Optimal Control Problem

Given a fixed choice for the tradeoff parameter C, find an error correction strategy $\{\vartheta_n^*, \zeta_n^*\}$ which minimizes $J_C[\{\vartheta_n, \zeta_n\}]$.

Can be solved using *quantum filtering* and *dynamic programming*.

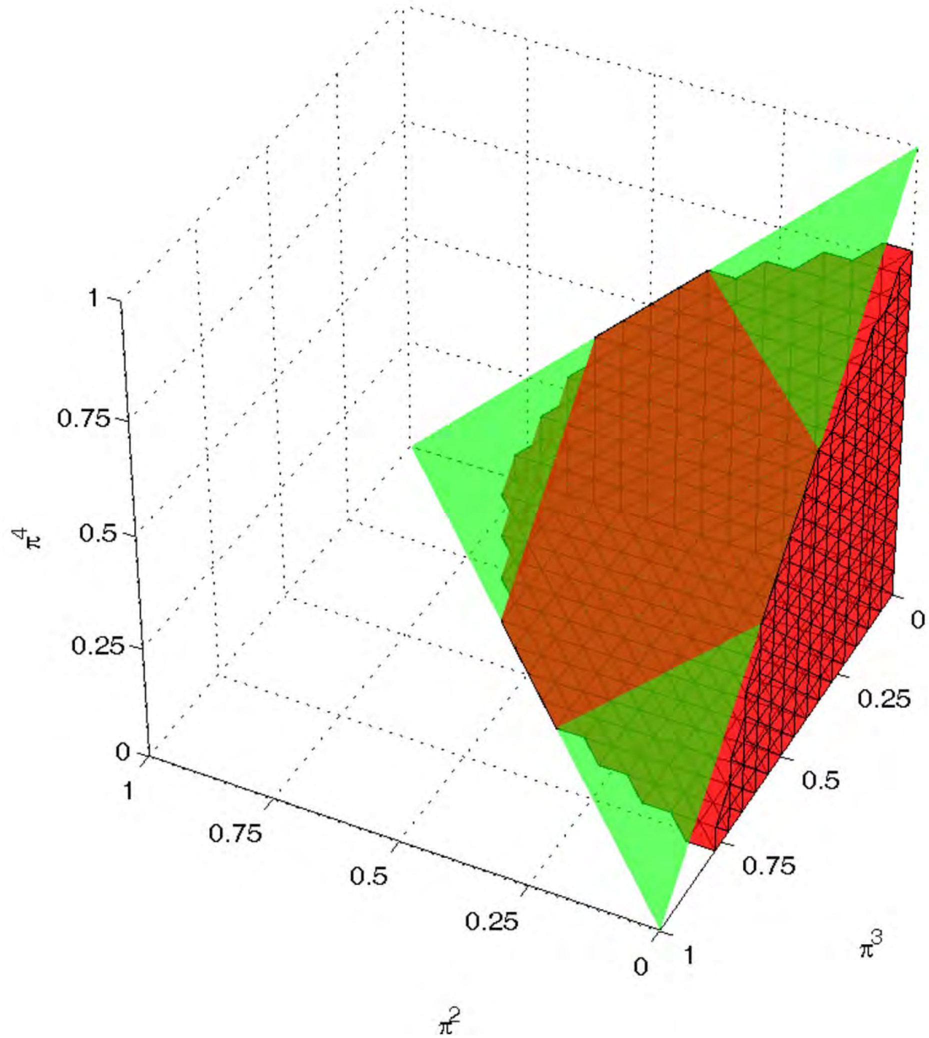
What does the optimal strategy look like? Separates into several steps:

1. First, the syndrome observations a **filtered**. The filter computes the conditional probabilities π_t^i of being in the ith syndrome at time t.
2. The space of all probabilities Π is partitioned into one **continuation region** Π_0 and a **correction region** Π_j for each possible correction.
3. The **optimal strategy**: we do nothing as long as $\pi_t \in \Pi_0$. As soon as π_t enters one of Π_j, $j \geq 1$ we perform the corresponding correction. This brings us back to Π_0, and the procedure repeats.

Numerical Solution

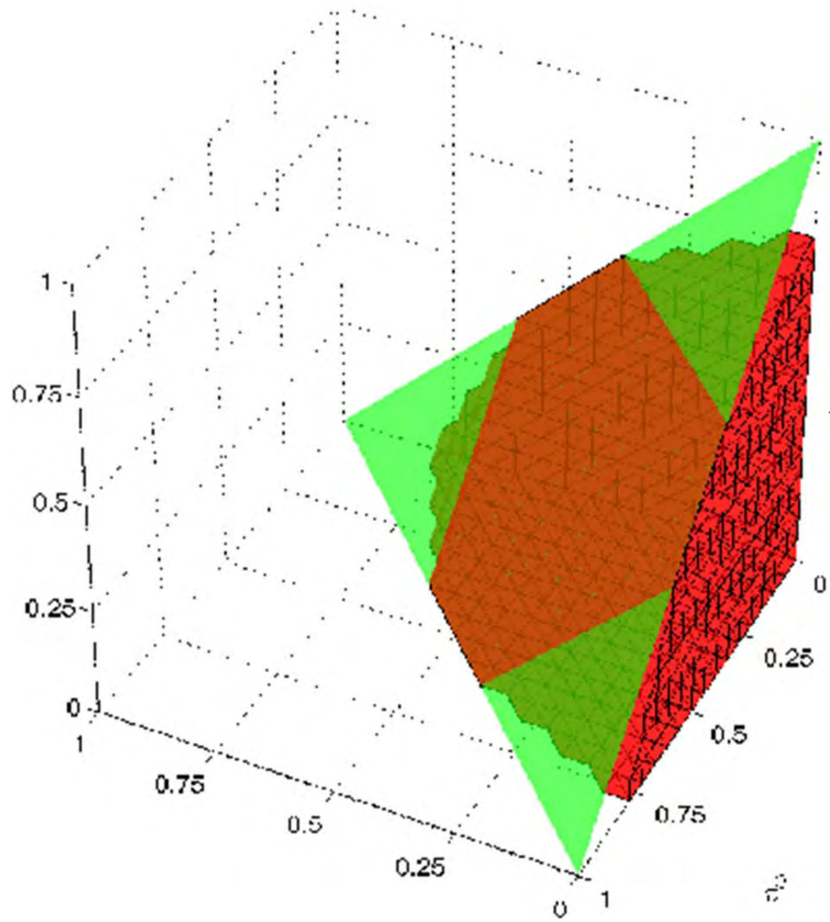
Finding the optimal strategy comes down to computing the continuation and correction regions Π_0, Π_j. This can be done numerically.

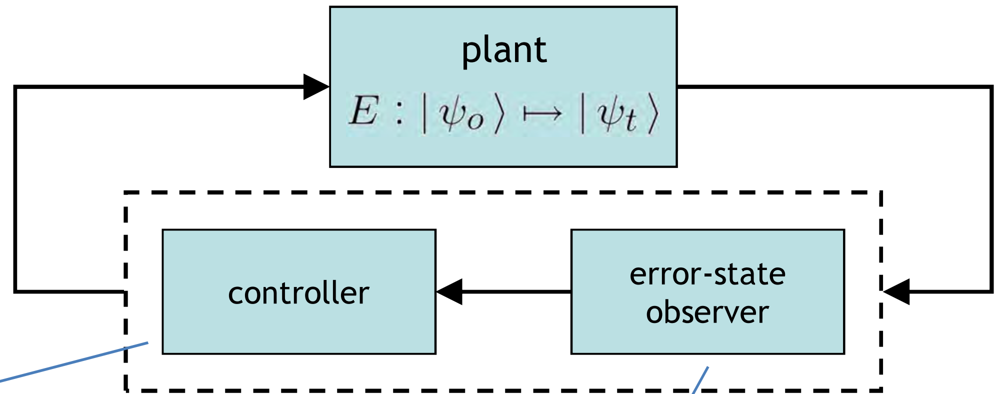
A simple three qubit code example (red region is Π_0):



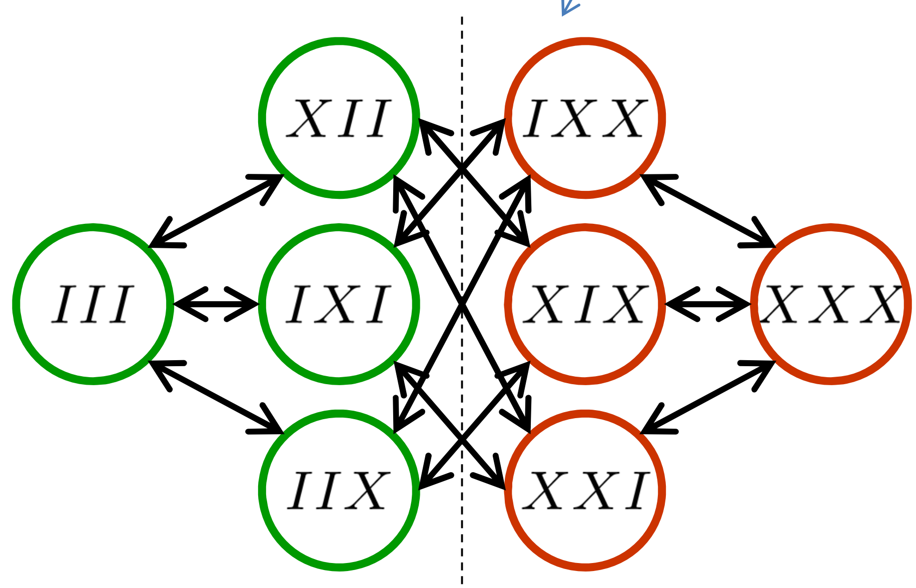
Quantum memory with separated control strategy

HM (and R. van Handel), New J. Phys. **11**, 105044 (2009)





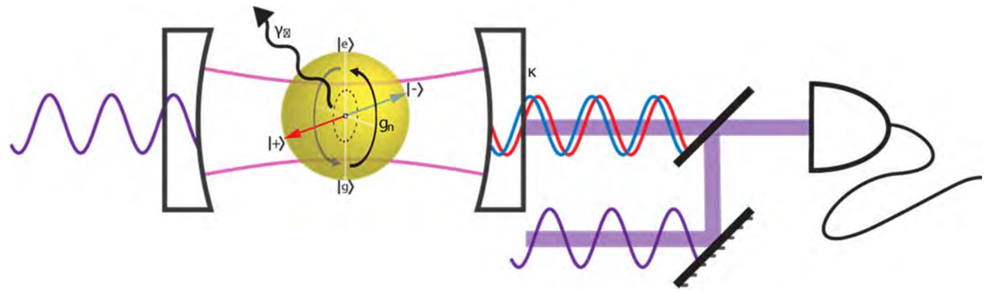
$$J_C[\{\vartheta_n, \zeta_n\}] = \mathbf{P}[\text{Wrong syndrome at time } \tau] + C\, \mathbf{E}[\#\{n : \vartheta_n \leq \tau\}].$$

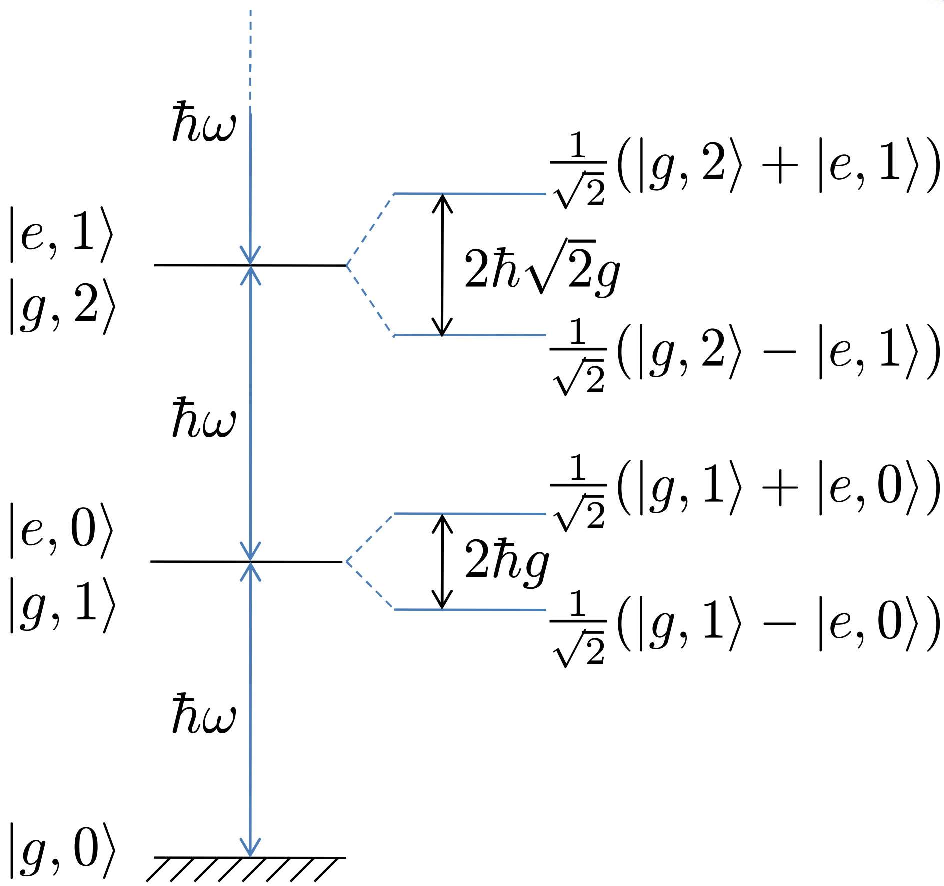


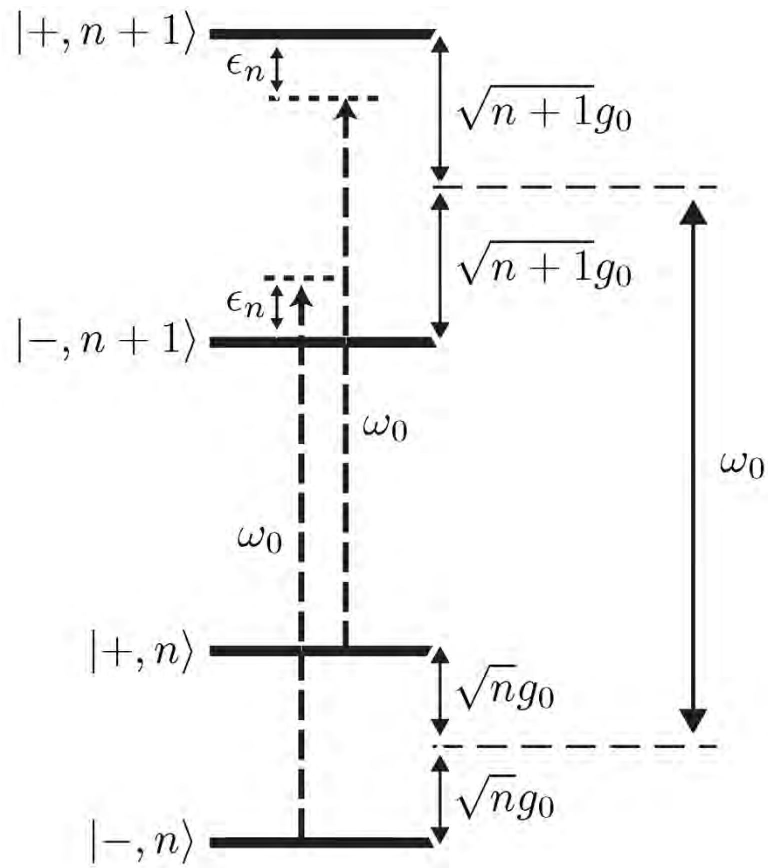
$$dp_j = (\textstyle\sum_{i\neq j} \nu_{ij} p_i - \nu_j p_j)dt + \sum_k \beta_k^{-2} p_j (a_{jk} - \langle M_k \rangle)(M_k - \langle M_k \rangle)$$

Spontaneous phase switching in cavity QED

P. Alsing and H. J. Carmichael, Quantum Opt. **3**, 13 (1991)

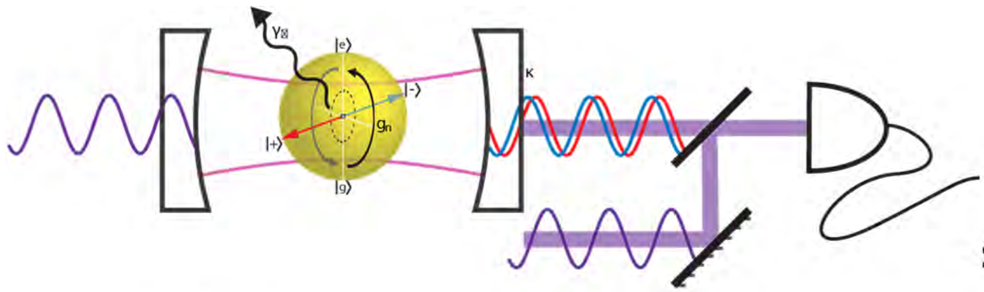


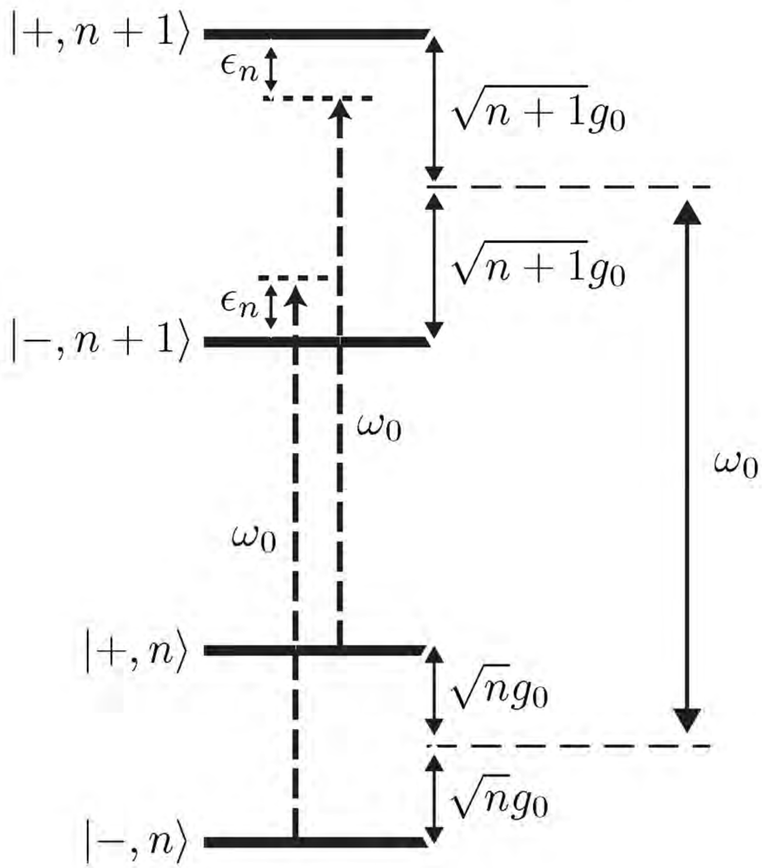


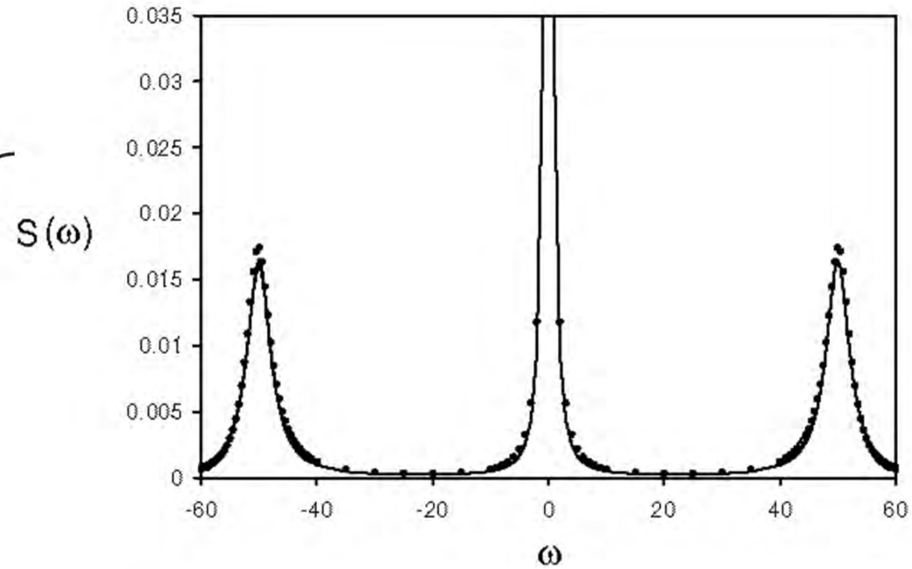


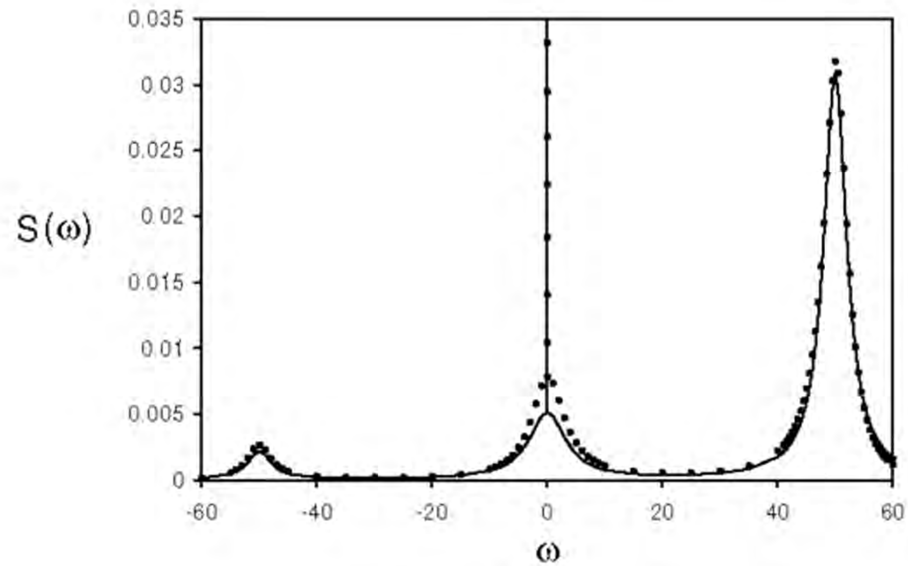
Feedback control: the Mollow doublet

J. E. Reiner, H. M. Wiseman and HM, Phys. Rev. A **67**, 042106 (2003)





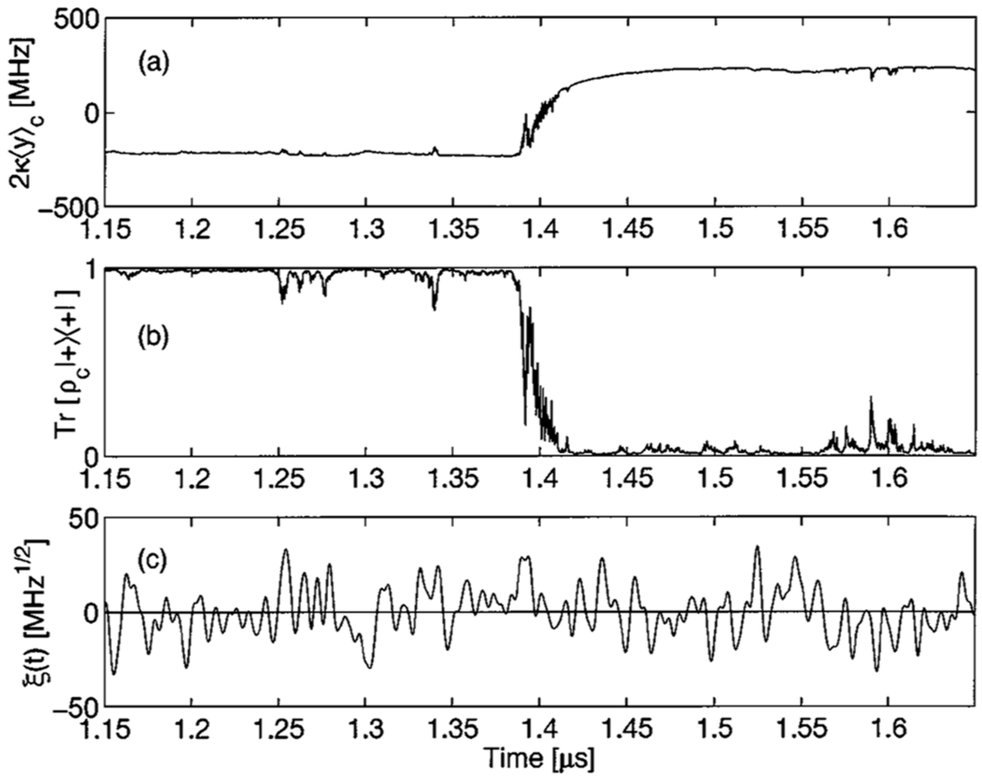


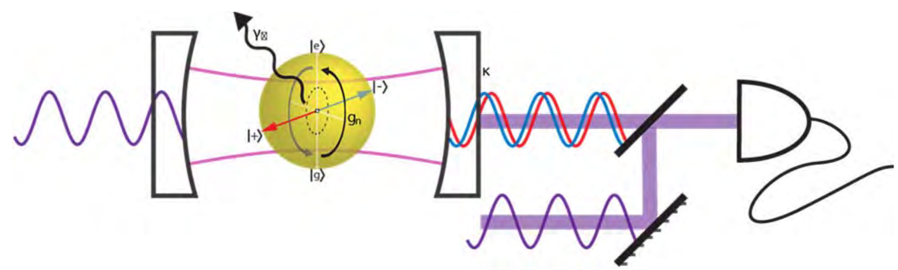


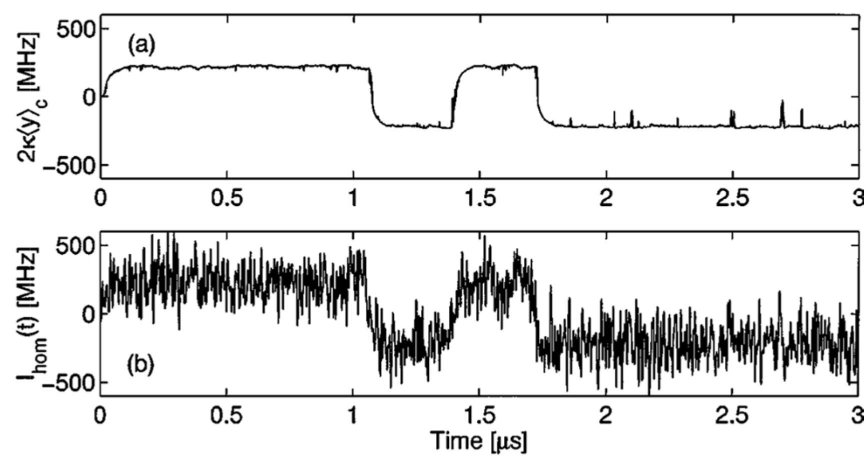
‘Retroactive’ quantum jumps

P. Alsing and H. J. Carmichael, Quantum Opt. **3**, 13 (1991)
HM and H. M. Wiseman, Phys. Rev. Lett. **81**, 4620 (1998)

$$
\begin{aligned}
d\rho &= \mathcal{L}\rho dt + i\sqrt{2\kappa\eta}\left\{a\rho - \rho a^{\dagger} - \mathrm{Tr}\left[\rho\left(a - a^{\dagger}\right)\right]\right\} dW_t \\
I_{\mathrm{hom}}(t)dt &= 2\eta\mathrm{Tr}\left[(-ia + ia^{\dagger})\rho\right] dt + \sqrt{2\kappa\eta}\, dW_t \\
dW_t &\equiv \left\{I_{\mathrm{hom}}(t)dt - 2\eta\mathrm{Tr}\left[(-ia + ia^{\dagger})\rho\right] dt\right\} / \sqrt{2\kappa\eta}
\end{aligned}
$$

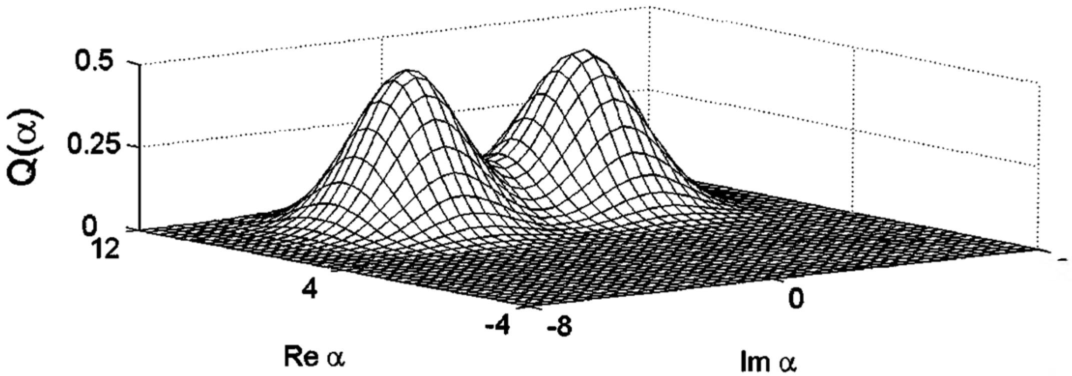


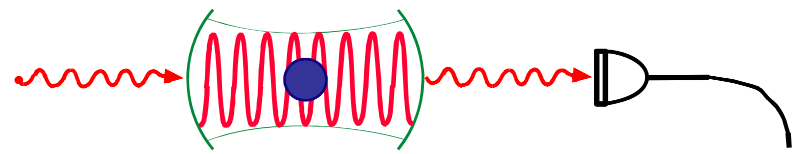


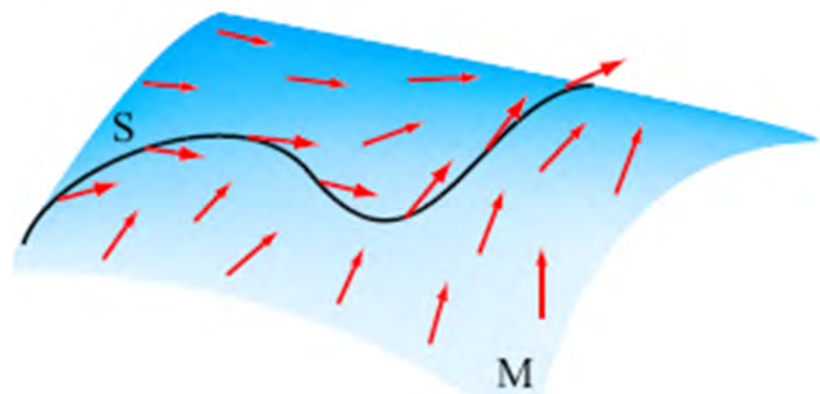


Phase bistability and quantum filter projection

P. Alsing and H. J. Carmichael, Quantum Opt. **3**, 13 (1991)
Ramon van Handel and HM, J. Opt. B: Quantum Semiclass. Opt. **7**, S226 (2005)
H. Mabuchi, Phys. Rev. A **78**, 015801, (2008)

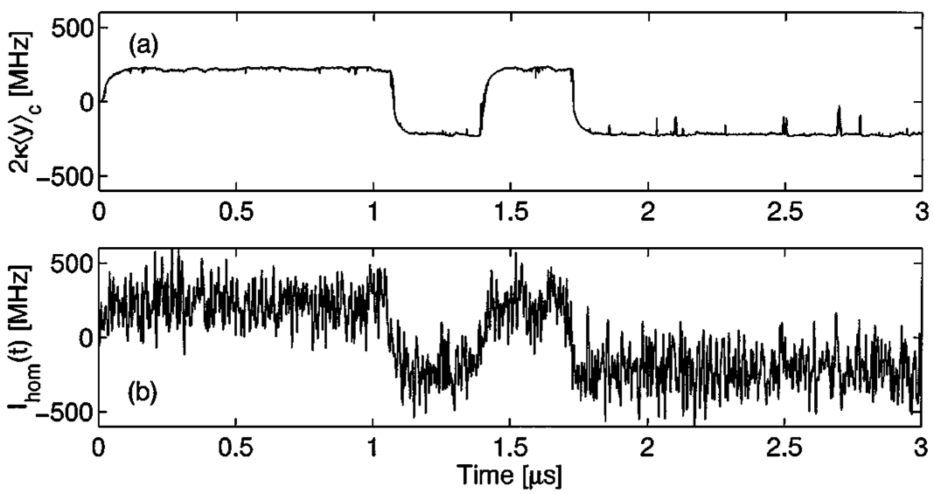






$$d\rho = \mathcal{L}\rho dt + i\sqrt{2\kappa\eta}\left\{a\rho - \rho a^\dagger - \mathrm{Tr}\left[\rho\left(a - a^\dagger\right)\right]\right\} dW_t$$

$$I_{\mathrm{hom}}(t) = 2\eta \mathrm{Tr}\left[(-ia + ia^\dagger)\rho\right] + \sqrt{2\kappa\eta}dW_t/dt$$



- parameterize sub-manifold of states
- intuition: two coupled “line segments”
- project stochastic equations of motion
- *e.g.*, Hilbert-Schmidt inner product

Bi-Gaussian approximate filter

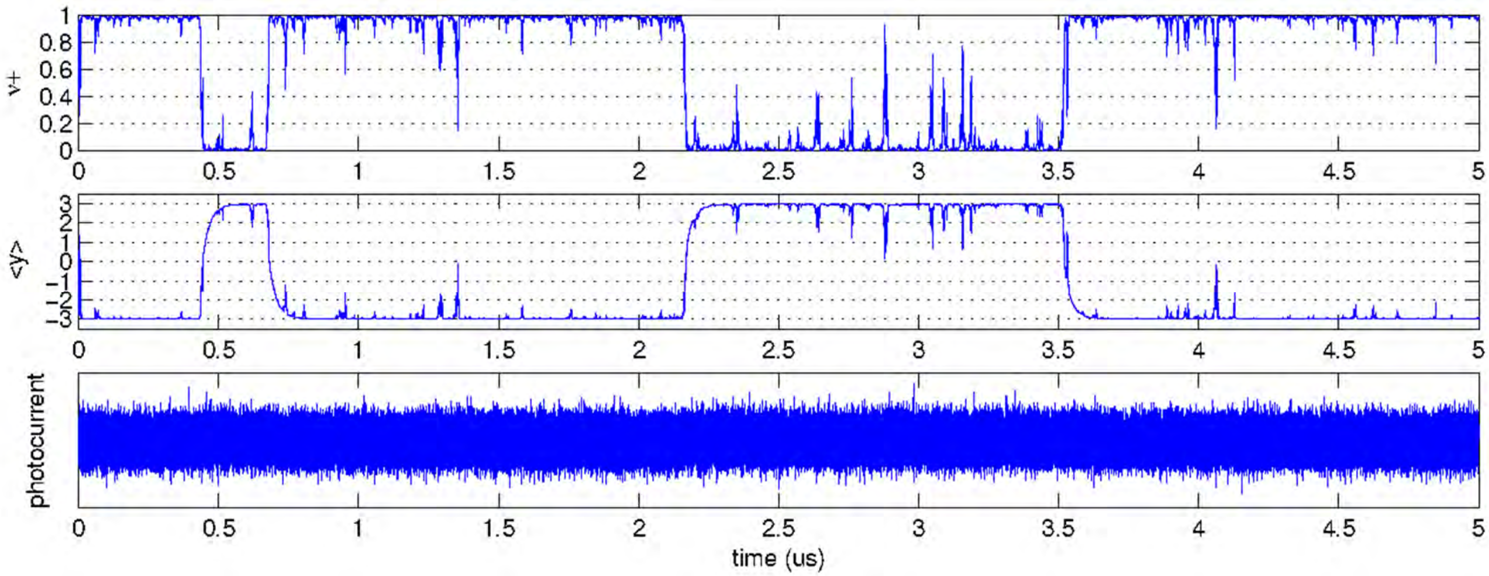
Ramon van Handel and HM, J. Opt. B: Quantum Semiclass. Opt. **7**, S226 (2005)

Physical intuition motivates Gaussian *ansatz*;
restriction by geometric methods *(D. Brigo et al., M. H. Vellekoop and J. M. C. Clark, …)*

$$d\tilde{\nu}_t^+ = -\gamma_\perp(\tilde{\nu}_t^+ - \tfrac{1}{2})dt + \sqrt{2\kappa\eta}\,\tilde{\nu}_t^+(1-\tilde{\nu}_t^+)(\mu_t^+ - \mu_t^-)(dY_t - \sqrt{2\kappa\eta}(\mu_t^+\tilde{\nu}_t^+ + \mu_t^-(1-\tilde{\nu}_t^+))dt)$$

$$\frac{d\mu_t^+}{dt} = -g - \kappa\mu_t^+ + \frac{\gamma_\perp}{2}\frac{1-\tilde{\nu}_t^+}{\tilde{\nu}_t^+}(\mu_t^- - \mu_t^+)$$

$$\frac{d\mu_t^-}{dt} = +g - \kappa\mu_t^- + \frac{\gamma_\perp}{2}\frac{\tilde{\nu}_t^+}{1-\tilde{\nu}_t^+}(\mu_t^+ - \mu_t^-)$$



accomplishes $\sim 10^5 \to 1$ reduction, but relies on knowing sub-manifold